Hyperbolic Coxeter polyhedra

Master's Thesis

Stepan A. Alexandrov

Advisor: prof. Nikolay V. Bogachev,
University of Toronto

Referee: prof. Andrei Yu. Vesnin,
Sobolev Institute of Mathematics

Moscow Institute of Physics and Technology
Moscow, 2023

Contents

1 Introduction **3**
- 1.1 Compact hyperbolic Coxeter polytopes . . . 3
  - 1.1.1 Classification of compact Coxeter products of simplices . . . 4
  - 1.1.2 Compact 3-free Coxeter polytopes . . . 5
- 1.2 Hyperbolic right-angled polyhedra . . . 5
  - 1.2.1 Dimension bounds for right-angled hyperbolic polyhedra . . . 5
  - 1.2.2 Number of ideal vertices of right-angled hyperbolic polyhedra . . . 6
- 1.3 Volumes of right-angled 3-polyhedra . . . 6
  - 1.3.1 Ideal right-angled hyperbolic 3-polyhedra . . . 7
  - 1.3.2 Compact right-angled hyperbolic 3-polytopes . . . 7
  - 1.3.3 Finite volume right-angled hyperbolic 3-polyhedra . . . 8
- Acknowledgements . . . 8

2 Preliminaries **9**
- 2.1 Abstract diagrams . . . 9
- 2.2 Coxeter diagrams . . . 9
- 2.3 Hyperbolic polytopes . . . 10
- 2.4 Hyperbolic Coxeter polytopes . . . 12
- 2.5 Superhyperbolic diagrams . . . 12
- 2.6 Hyperbolic right-angled polyhedra . . . 14
- 2.7 Combinatorics of Euclidean polytopes . . . 14
- 2.8 Combinatorics of ideal right-angled hyperbolic 3-polyhedra . . . 15
- 2.9 Combinatorics of compact right-angled hyperbolic 3-polytopes . . . 16
- 2.10 Combinatorics of finite volume right-angled hyperbolic 3-polyhedra . . . 18

3 Proofs **20**
- 3.1 Proof of Theorems 1.1 and 1.2 . . . 20
- 3.2 Proof of Theorem 1.3 . . . 31
- 3.3 Proof of Theorem 1.5 . . . 32
- 3.4 Proof of Theorem 1.6 . . . 33
- 3.5 Proof of Theorem 1.8 . . . 34
- 3.6 Proof of Theorem 1.9 . . . 35
- 3.7 Proof of Theorem 1.10 . . . 38
- 3.8 Proof of Theorem 1.11 . . . 38

Author's papers **41**

References **41**

Chapter 1

Introduction

Let $\mathbb{X}^d$ denote a simply connected space of constant sectional curvature, i.e. either spherical space $\mathbb{S}^d$, or Euclidean space $\mathbb{E}^d$, or hyperbolic space $\mathbb{H}^d$. A convex finite volume polyhedron $P \subset \mathbb{X}^d$ is called a *Coxeter polyhedron* if its dihedral angles are all integer submultiples of π. One can proof that the group $\Gamma(P) < \mathrm{Isom}(\mathbb{X}^d)$ generated by reflections in facets of P is discrete. It turns out that this fact holds only for Coxeter polyhedra. Thus, the study of reflective (cocompact) lattices in $\mathrm{Isom}(\mathbb{X}^d)$ is no different from the study of (compact) Coxeter polyhedra in $\mathbb{X}^d$.

Compact Coxeter polytopes in $\mathbb{S}^d$ and $\mathbb{E}^d$ were classified by Coxeter in [Cox34]. Vinberg initiated the study of such polytopes in $\mathbb{H}^d$ and proved in [Vin84] that there are no compact Coxeter polytopes in $\mathbb{H}^{\geqslant 30}$. Examples are known only in $\mathbb{H}^{\leqslant 8}$, a single example is known in $\mathbb{H}^8$ and two in $\mathbb{H}^7$ ([Bug92, Bug84, All13]). Finite volume polyhedra of this type do not exist in $\mathbb{H}^{\geqslant 996}$ (see [Kho86, Pro86]). Examples are known only in $\mathbb{H}^{\leqslant 19}$ and $\mathbb{H}^{21}$ (see [Vin72, KV78, Bor87]) In the course of our research, we found that compact hyperbolic Coxeter polytopes have certain restrictions (Theorem 1.1). These restrictions imply the classification of compact polytopes that combinatorially equivalent to products of simplices (Theorem 1.2). We also succeed to improve the Vinberg upper bound on the dimensions for a certain type of polytopes (Theorem 1.3).

A special case of Coxeter polyhedra are *right-angled polyhedra*, i.e. convex polyhedra with all dihedral angles being equal to $\pi/2$. It is possible to improve the upper bound on the dimensions of such polyhedra (see Theorems 1.4, 1.5, and 1.6). We propose an improvement of the Nikulin inequality, which is widely used in proving the absence of Coxeter polyhedra. This allows us to give much shorter proofs for the second and third theorems. During the search of the further improvements on the dimension we managed to improve Nonaka's lower bound on the number of ideal vertices (see Theorem 1.8).

Finally, in our joint work with N. Bogachev, A. Vesnin, and A. Egorov, we consider 3-dimensional right-angled polyhedra. Recently, Champanerkar, Kofman, and Purcell conjectured that a knot complement does not admit a decomposition into ideal hyperbolic right-angled polyhedra (see [CKP22]). They verified the conjecture for all knots up to 11 crossings by comparing their volumes with the smallest volumes of ideal right-angled polyhedra. We studied the volumes of polyhedra and improved Atkinson's lower volume bound (Theorems 1.9, 1.10, and 1.11).

1.1 Compact hyperbolic Coxeter polytopes

There are two very hard long-standing open problems in the theory of compact hyperbolic Coxeter polytopes. The first one is the construction of new hyperbolic Coxeter polytopes, especially higher-dimensional ones. And the second one is the classification of such polytopes.

Generally speaking, there are two different approaches to both problems: classification of finite volume Coxeter polytopes of some certain combinatorial types (see [Kap74, Ess96, Tum07, FT08, FT09, JT18, Bur22, MZ22a, MZ22b]) and the theory of arithmetic hyperbolic reflection groups (see [Vin72, Bel16, Bog17, BP18, Bog19, Bog20]). In particular, in the context of arithmetic and quasi-arithmetic reflection groups several authors constructed new Coxeter polytopes as faces or reflection centralizers of some higher dimensional polytopes (see [Bor87, All06, All13, BK21, BBKS21]).

Our work was focused on the combinatorial approach, so let us give a brief summary of the results on the classification of compact hyperbolic Coxeter polytopes of certain combinatorial properties. The complete classification of Coxeter polytopes in $\mathbb{H}^2$ was obtained by Poincaré ([Poi82]). Andreev ([And70a, And70b]) described all Coxeter polytopes in $\mathbb{H}^3$. Compact Coxeter simplices were classified by Lannér ([Lan50]). Kaplinskaya ([Kap74]) used this classification to list all compact simplicial prisms. Esselmann ([Ess96]) used Gale diagrams to list the remaining compact polytopes in $\mathbb{H}^d$ with $d+2$ facets. Tumarkin ([Tum07]) improved this technique and listed all compact polytopes in $\mathbb{H}^d$ with $d+3$ facets. All cubes were classified by Jacquemet and Tschantz ([JT18]). Very recently and independently, Burcroff ([Bur22]) and Ma & Zheng ([MZ22a, MZ22b]) listed all compact Coxeter polytopes in $\mathbb{H}^d$ with $d+4$ facets for $d = 4, 5$.

1.1.1 Classification of compact Coxeter products of simplices

First of all, we should provide some definitions. Each Coxeter polytope can be described by its Coxeter diagram. Such a diagram contains information about the dihedral angles and distances between every pair of facets. Another way to describe a Coxeter polytope is its Gram matrix, i.e. the matrix of inner products of the outward normal vectors to the facets. Since such vectors are in Minkowski space, the negative inertia index of the Gram matrix of a hyperbolic Coxeter polytope is equal to one. Both descriptions are equivalent: one can obtain the Gram matrix of a Coxeter polytope using its Coxeter diagram, and vice versa. It is known that the Coxeter diagram of a compact Coxeter polytope does not contain a parabolic subdiagram, i.e. a diagram whose Gram matrix is positive semi-definite and singular.

Coxeter diagrams of the compact simplices in hyperbolic spaces were listed by Lannér ([Lan50]) and are now called Lannér diagrams. They have an important property. Consider a compact hyperbolic Coxeter polytope and a minimal set of its facets with empty intersection. The subdiagram that corresponds to the set is a Lannér diagram.

The Lannér diagrams play an important role in many classifications as such diagrams are "unfriendly" to each other. These diagrams often form so-called superhyperbolic diagrams, i.e. the diagrams whose negative inertia index of the corresponding Gram matrix is at least two. Such diagrams are not contained in any diagram that corresponds to a hyperbolic Coxeter polytope. Our first theorem provides a result of this type.

Denote by $\mathcal{L}_{k_1} \times_0 \cdots \times_0 \mathcal{L}_{k_n}$ the set of all Coxeter diagrams generated by pairwise disjoint Lannér diagrams of orders $k_1, \dots, k_n$ and not containing parabolic or other Lannér subdiagrams. The Gram matrices of such diagrams can be characterised as follows. They are symmetric. The diagonal consists of blocks that are the Gram matrices of Lannér diagrams of orders $k_1, \dots, k_n$. Each of the other values is either equal to $-\cos\left(\frac{\pi}{k}\right)$, or a real not exceeding -1. The Gram matrix does not have the Gram matrix of a Lannér or parabolic diagram as its principal submatrix, except for those that are on the diagonal.

Let us introduce the notation for some families of compact hyperbolic Coxeter polytopes:

- Simp^* for all products of simplices;
- Simp^k for all products of k simplices;
- Cubes for all cubes (not necessarily 3-dimensional).

Theorem 1.1 ([Ale22, Theorem A]). *Let $n \geqslant 4$ and $2 \neq k_1 \geqslant \cdots \geqslant k_n = 2$. Every diagram contained in the set $\mathcal{L}_{k_1} \times_0 \cdots \times_0 \mathcal{L}_{k_n}$ is superhyperbolic.*

As a simple corollary of this theorem, we obtain the following.

Theorem 1.2 ([Ale22, Theorem B]). $\mathsf{Simp}^* = \mathsf{Simp}^1 \cup \mathsf{Simp}^2 \cup \mathsf{Simp}^3 \cup \mathsf{Cubes}$.

Recall that 2-dimensional hyperbolic polytopes were studied by Poincaré in [Poi82]. Acute-angled 3-dimensional hyperbolic polytopes have a good combinatorial description by Andreev ([And70a, And70b]). Higher-dimensional polytopes are listed by:

- Simp^1: Lannér ([Lan50]);

- Simp^2: Kaplinskaya ([Kap74]) and Esselmann ([Ess96]);
- Simp^3: Tumarkin ([Tum07]);
- Cubes: Jacquemet and Tschantz ([JT18]).

Thus, the theorem provides the complete classification of the compact hyperbolic Coxeter polytopes that are combinatorially equivalent to products of simplices.

1.1.2 Compact 3-free Coxeter polytopes

Now let us consider the polytopes with diagram containing no Lannér subdiagrams of order $\geqslant 3$. These are exactly the polytopes with the following property: every set of facets with an empty intersection contains a pair of disjoint facets. Such polytopes are called 3-free polytopes. For example, cubes satisfy this property, so the Coxeter diagram of a cube does not contain a Lannér subdiagram of order $\geqslant 3$. Another example that satisfies this property is the family of compact right-angled polytopes in hyperbolic spaces (the reason is the structure of their diagrams). It is known that there are no Coxeter cubes in $\mathbb{H}^{\geqslant 6}$ ([JT18]) and that there are no compact right-angled polytopes in $\mathbb{H}^{\geqslant 5}$ ([PV05]). Recently Burcroff in [Bur22] used Vinberg's methods to estimate the dimension of such polytopes. We slightly improved this estimation.

Theorem 1.3 ([Ale22, Theorem C])**.** *Every Coxeter diagram of a compact Coxeter polytope in $\mathbb{H}^{\geqslant 13}$ contains a Lannér diagram of order $\geqslant 3$.*

1.2 Hyperbolic right-angled polyhedra

Unlike in spherical or Euclidean spaces, the combinatorics of right-angled polytopes in hyperbolic space is more complex. For example, one may start with a regular right-angled dodecahedron in $\mathbb{H}^3$. Then one can glue two such dodecahedra together by identifying a pair of mutually isometric pentagonal faces and obtain a new right-angled polytope. This procedure can be performed inductively to obtain a garland of dodecahedra. In the Euclidean case, the only right-angled polytope is a parallelogram, and any such garland would be just a parallelogram again.

1.2.1 Dimension bounds for right-angled hyperbolic polyhedra

Since right-angled polyhedra are also Coxeter polyhedra, the Vinberg–Khovanskii–Prokhorov dimension upper bounds also holds for them. However, these bounds can be significantly improved as follows.

Theorem 1.4 ([PV05])**.** *There are no compact right-angled polyhedra in $\mathbb{H}^{\geqslant 5}$.*

Our next result is a new short proof of the following two theorems.

Theorem 1.5 ([Kol12])**.** *There are no ideal right-angled polyhedra in $\mathbb{H}^{\geqslant 7}$.*

Theorem 1.6 ([PV05, Duf10])**.** *There are no finite volume right-angled polyhedra in $\mathbb{H}^{\geqslant 13}$.*

The bound stated in Theorem 1.4 is exact. The exactness of the other bounds is unknown: examples of ideal right-angled polyhedra are only known up to dimension 4 and examples of finite volume right-angled polyhedra are only known up to dimension 8.

Nikulin's inequality ([Nik81, Theorem 3.2.1]) states that low-dimensional faces of a simple Euclidean polytope cannot have too many faces on average. Khovanskii proved that Nikulin's inequality holds for polytopes that are simple at edges ([Kho86, Theorem 10]). We prove that Nikulin's inequality not only holds for polytopes that are simple at edges but can also sometimes be improved. Section 3.3 deals with the inequality for 7-dimensional ideal finite volume right-angled hyperbolic polyhedra and Section 3.4 deals with the inequality for 13-dimensional right-angled hyperbolic polyhedra. Our proofs are based on the fact that every ideal vertex is contained in many facets. We believe that a more general case can be proved.

1.2.2 Number of ideal vertices of right-angled hyperbolic polyhedra

Lower bounds on the number of ideal vertices and facets of right-angled polyhedra are essential to prove the absence of such polyhedra in higher dimensions. Let $\mathcal{P}^n$ denote the family of finite volume non-compact right-angled hyperbolic polyhedra, $a_k(P)$ and $v_\infty(P)$ denote the number of k-faces and the number of ideal vertices of a polyhedron P respectively. In [Non15] Nonaka obtained some estimates on the number of ideal vertices of a right-angled hyperbolic polyhedron.

Theorem 1.7 ([Non15, Main Theorem 1.2]). *Let $P^n \in \mathcal{P}^n$. Then $v_\infty(P^n) \geqslant v_\infty^n$, where v_∞^n is defined by the following table.*

n	5	6	7	8	9	10	11	12
v_∞^n	–	3	17	36	91	254	741	2200

We managed to improve Nonaka's estimates as follows.

Theorem 1.8 ([Ale23, Theorem 1.5]). *Let $P^n \in \mathcal{P}^n$. Then $v_\infty(P^n) \geqslant v_\infty^n$, where v_∞^n is defined by the following table.*

n	5	6	7	8	9	10	11	12
v_∞^n	2	6	23	135	1704	182 044	$1.67 \cdot 10^9$	$1.27 \cdot 10^{17}$

The proof of the theorem and more precise bounds are given in Section 3.5.

1.3 Volumes of right-angled 3-polyhedra

Studying volumes of hyperbolic polyhedra and hyperbolic manifolds is a fundamental problem in geometry and topology. Recently, the study of the volumes of ideal right-angled polyhedra has become more attractive and important in view of the theorem on maximal volume of a generalized hyperbolic polyhedra with given 1-skeleton [Bel21], the problem on minimal ideal right-angled 4-dimensional hyperbolic polyhedron [Kol12], and the conjecture about hyperbolic right-angled knots [CKP22].

An initial list of ideal right-angled polyhedra is given in [EV20a], and of compact ones in [Ino22]. A detailed discussion of constructions of hyperbolic 3-manifolds from right-angled polyhedra can be found in the recent survey [Ves17]. In the compact case, there such manifolds are related to small covers, see e.g. [DJ91, BP16]. Also, right-angled polytopes are useful for constructing hyperbolic 3-manifolds that bound geometrically [KMT15]. Let us also mention several works on the interplay between the arithmeticity of hyperbolic reflection groups and arithmeticity of hyperbolic links [Kel22, MMT20]. Here it turns out to be very useful that fundamental groups of some hyperbolic link complements are commensurable with hyperbolic reflection groups since Vinberg's theory of reflection groups [Vin85] can be applied.

In 1970, Andreev [And70a, And70b] (see also [RHD07]) obtained his famous characterization of hyperbolic acute-angled (all dihedral angles are at most $\pi/2$) 3-polyhedra of finite volume. For right-angled (all dihedral angles equal $\pi/2$) polyhedra, Andreev's theorems provide simple necessary and sufficient conditions for realizing a given combinatorial type as a compact, ideal, or finite volume polyhedron in $\mathbb{H}^3$. Such realizations are determined uniquely up to isometry of $\mathbb{H}^3$. Thus one can expect that geometric invariants of these polyhedra can be estimated via combinatorics. Lower and upper bounds for volumes of right-angled hyperbolic polytopes using the number of vertices were obtained by Atkinson [Atk09].

Recall that volumes of hyperbolic 3-polyhedra can usually be expressed via the *Lobachevsky function* (see [Vin93b])

$$\Lambda(x) = -\int_0^x \log|2\sin t|dt.$$

In order to formulate the main results more conveniently, we define two constants depending only on Lobachevsky function's values at certain points. The first one, $v_8 = 8\Lambda(\pi/4)$, equals the volume of the regular ideal hyperbolic octahedron. Up to six decimal places $v_8 \approx 3.663862$. The second one, $v_3 = 3\Lambda(\pi/3)$, equals the volume of the regular ideal hyperbolic tetrahedron. Up to six decimal places $v_3 \approx 1.014941$.

1.3.1 Ideal right-angled hyperbolic 3-polyhedra

Recall that if $P \subset \mathbb{H}^3$ is an ideal right-angled polyhedron with V vertices, then $V \geqslant 6$. Moreover, $V = 6$ if and only if P is an octahedron, which can be described as the antiprism $A(3)$ with triangular bases. Thus, $\mathrm{Vol}(A(3)) = v_8$.

The volume formula for antiprisms $A(n)$ (i.e. ideal right-angled polytopes with $V = 2n$ vertices, two n-gonal bases and $2n$ lateral triangles), $n \geqslant 3$, was obtained by Thurston [Thu80, Chapter 6 & 7]:

$$\mathrm{Vol}(A(n)) = 2n \left[\Lambda\left(\frac{\pi}{4} + \frac{\pi}{2n}\right) + \Lambda\left(\frac{\pi}{4} - \frac{\pi}{2n}\right)\right].$$

For example, up to six decimal places, $\mathrm{Vol}(A(4)) = 6.023046$.

In 2009, Atkinson obtained [Atk09, Theorem 2.2] the following upper and lower bounds for volumes via the number of vertices. Let P be an ideal right-angled hyperbolic 3-polytope with $V \geqslant 6$ vertices, then

$$\frac{v_8}{4} \cdot V - \frac{v_8}{2} \leqslant \mathrm{Vol}(P) \leqslant \frac{v_8}{2} \cdot V - 2v_8.$$

It is worth mentioning that both inequalities are sharp when P is a regular ideal octahedron (i.e. for $V = 6$). Moreover, the upper bound is asymptotically sharp in the following sense: there exists a sequence of ideal right-angled polytopes P_i with V_i vertices such that $\mathrm{Vol}(P_i)/V_i \to \frac{v_8}{2}$ as $i \to +\infty$.

There are no ideal right-angled polytopes with $V = 7$, and $V = 8$ if and only if P is the antiprism $A(4)$ with quadrilateral bases. The following upper bound was obtained in [EV20c, Theorem 2.2]. Let P be an ideal right-angled hyperbolic 3-polyhedron with $V \geqslant 9$ vertices. Then

$$\mathrm{Vol}(P) \leqslant \frac{v_8}{2} \cdot V - \frac{5v_8}{2}.$$

The inequality is sharp when P is the double of a regular ideal octahedron along a face (i.e. for $V = 9$). The graphs of the above lower and upper bounds in comparison to the volumes of ideal right-angled polyhedra up to 21 vertices can be found in [EV20c, Fig. 1].

Theorem 1.9 ([ABVE23, Theorem 1.1])**.** *Let P be an ideal right-angled hyperbolic 3-polyhedron with V vertices. Then the following inequalities hold.*

(1) If $V > 24$, then
$$\mathrm{Vol}(P) \leqslant \frac{v_8}{2} \cdot V - 3v_8.$$

(2) If P has a k-gonal face, $k \geqslant 3$, then
$$\mathrm{Vol}(P) \leqslant \frac{v_8}{2} \cdot V - \frac{k+5}{4} v_8.$$

(3) If P has only triangular and quadrilateral faces with $V \geqslant 73$, then
$$\mathrm{Vol}(P) \leqslant \frac{v_8}{2} \cdot V - (9v_8 - 20v_3).$$

The proof of Theorem 1.9 is given in Section 3.6.

1.3.2 Compact right-angled hyperbolic 3-polytopes

It is well-known that for a compact right-angled polytope in $\mathbb{H}^3$ with V vertices we have either $V = 20$ or $V \geqslant 24$ and even. Moreover, $V = 20$ if and only if P is a regular right-angled dodecahedron.

The volume formula is known for an infinite series of compact right-angled *Löbell polytopes* $L(n)$, $n \geqslant 5$, with $V = 4n$, two n-gonal bases and $2n$ lateral pentagonal faces. In particular, $L(5)$ is a regular dodecahedron. By [Ves98], for $n \geqslant 5$, the volume of $L(n)$ is

$$\mathrm{Vol}(L(n)) = \frac{n}{2}\left[2\Lambda(\theta) + \Lambda\left(\theta + \frac{\pi}{n}\right) + \Lambda\left(\theta - \frac{\pi}{n}\right) - \Lambda\left(2\theta - \frac{\pi}{2}\right)\right],$$

where $\theta = \frac{\pi}{2} - \arccos\left(\frac{1}{2\cos(\pi/n)}\right)$.

Two-sided bounds for volumes of compact right-angled hyperbolic 3-polytopes were obtained by Atkinson [Atk09, Theorem 2.3]. Namely, if P is a compact right-angled hyperbolic 3-polytope with V vertices, then

$$\frac{v_8}{32} \cdot V - \frac{v_8}{4} \leqslant \mathrm{Vol}(P) < \frac{5v_3}{8} \cdot V - \frac{25}{4} v_3.$$

Moreover, there exists a sequence of compact right-angled polytopes P_i with V_i vertices such that $\mathrm{Vol}(P_i)/V_i \to \frac{5v_3}{8}$ as $i \to +\infty$.

The upper bound can be improved if we exclude the case of the dodecahedron. Indeed, by [EV20c, Theorem 2.4], if P is a compact right-angled hyperbolic 3-polytope with $V \geqslant 24$ vertices, then

$$\mathrm{Vol}(P) \leqslant \frac{5v_3}{8} \cdot V - \frac{35}{4} v_3.$$

Theorem 1.10 ([ABVE23, Theorem 1.2])**.** *Let P be a compact right-angled hyperbolic polytope with V vertices. Then the following inequalities hold.*

(1) If $V > 80$, then

$$\mathrm{Vol}(P) \leqslant \frac{5v_3}{8} \cdot V - 10v_3.$$

(2) If P has a k-gonal face, $k \geqslant 5$, then

$$\mathrm{Vol}(P) \leqslant \frac{5v_3}{8} \cdot V - \frac{5k+35}{8} v_3.$$

The proof of Theorem 1.10 is given in Section 3.7.

1.3.3 Finite volume right-angled hyperbolic 3-polyhedra

As well as in the ideal and compact cases, Atkinson obtained (see [Atk09, Theorem 2.4]) volume bounds for right-angled hyperbolic polyhedra having both finite and ideal vertices. If P is a finite volume right-angled hyperbolic polyhedron with $V_\infty > 0$ ideal vertices and V_F finite vertices, then

$$\frac{v_8}{4} \cdot V_\infty + \frac{v_8}{32} \cdot V_F - \frac{v_8}{4} \leqslant \mathrm{Vol}(P) < \frac{v_8}{2} \cdot V_\infty + \frac{5v_3}{8} \cdot V_F - \frac{v_8}{2}. \tag{1.1}$$

Provided more combinatorial information about P, we are able to improve the upper bound as follows.

Theorem 1.11 ([ABVE23, Theorem 1.3])**.** *Let P be a finite volume right-angled hyperbolic 3-polyhedron with $V_\infty > 0$ ideal vertices and V_F finite vertices. If $V_\infty + V_F > 17$, then*

$$\mathrm{Vol}(P) < \frac{v_8}{2} \cdot V_\infty + \frac{5v_3}{8} \cdot V_F - \left(v_8 + \frac{5v_3}{2}\right).$$

The proof of Theorem 1.11 is given in Section 3.8.

Acknowledgements

The author is grateful to Nikolay Bogachev for his supervision and helpful discussions, to Anna Felikson for maintaining her webpage on hyperbolic Coxeter polytopes, and to Alexander Kolpakov, Andrei Vesnin, Andrei Egorov, Ignat Soroko, and the referees for their remarks.

Chapter 2

Preliminaries

2.1 Abstract diagrams

A *diagram* is a graph with positive real weights on the edges. The *order* $|S|$ of a diagram S is the number of vertices of the graph. A *subdiagram* of a diagram S is a diagram obtained from S by erasing some vertices together with all edges incident to these vertices. Consider a diagram S. A diagram *generated* by subdiagrams $S_1, \dots, S_k$ of S and vertices $v_1, \dots, v_l$ of S is a subdiagram $\langle S_1, \dots, S_k, v_1, \dots, v_l \rangle$ of S obtained from S by erasing every vertex v that is not contained in any S_i and is not equal to any v_j.

Let S be a diagram. Consider a symmetric matrix (g_{ij}) such that g_{ij} equals one if $i = j$, zero if $v_i v_j$ is not an edge of the diagram S, and $-w_{ij}$ if w_{ij} is the weight of the edge $v_i v_j$. Such a matrix $G(S) = (g_{ij})$ is called the *Gram matrix* of the diagram S.

We say that a diagram has some property if its Gram matrix has the same property (e.g., positive definiteness). A diagram has the same determinant and signature as its Gram matrix.

A diagram is said to be *elliptic* if it is positive definite, *parabolic* if it is positive semidefinite and not elliptic, and *hyperbolic* if it is indefinite with the negative inertia index equals one.

A *product* of diagrams S_1 and S_2 is a diagram whose vertex set is the disjoint union of the vertex sets of S_1 and S_2 and whose edge set is the union of the edge sets of S_1 and S_2 (informally speaking, we draw two diagrams side by side). The Gram matrix of such diagram is equal to $G(S_1) \oplus G(S_2)$ up to simultaneous permutation of rows and columns. A diagram is *connected* if it is not a product of some other non-empty diagrams.

Obviously, every elliptic diagram is a product of some connected elliptic diagrams. Every parabolic diagram is a product of some connected elliptic diagrams and some (at least one) connected parabolic diagrams.

Proposition 2.1. *A hyperbolic diagram does not contain a subdiagram that is a product of two hyperbolic diagrams.*

2.2 Coxeter diagrams

A diagram is called a *Coxeter diagram* if each of its weights is either $\geqslant 1$ or equal to $\cos\left(\frac{\pi}{m}\right)$ for some integer $m \geqslant 3$. Such diagrams are usually drawn as follows. If the weight of an edge $v_i v_j$ is greater than one, then a dashed edge is drawn connecting v_i and v_j. If the weight of an edge $v_i v_j$ is equal to one, then a bold edge is drawn. If the weight of an edge $v_i v_j$ is equal to $\cos(\frac{\pi}{m})$, then a $(m-2)$-fold edge or a simple edge with label m is drawn. We say that a vertex v is *joined* with a vertex u if they are joined by any edge other than a 2-labeled one.

Theorem 2.2 ([Cox34]). *Connected elliptic and parabolic diagrams are listed in Table 2.1 and Table 2.2, respectively.*

Corollary 2.3. *Every elliptic diagram contains no cycle. Every vertex of an elliptic diagram is joined with at most three other vertices.*

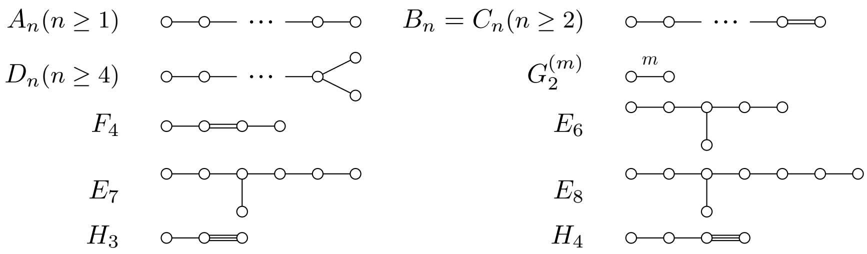


Table 2.1: Connected elliptic Coxeter diagrams

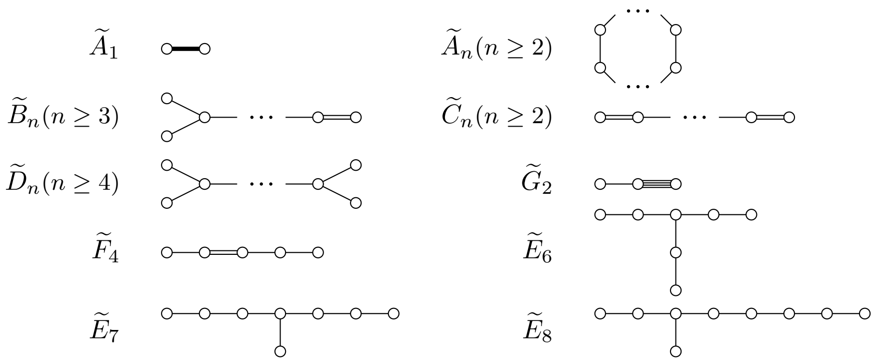


Table 2.2: Connected parabolic Coxeter diagrams

Corollary 2.4. *Let us decrease the multiplicities of some edges of an elliptic diagram. The obtained diagram is elliptic.*

Corollary 2.5. *Let us decrease the multiplicities of some edges of a parabolic diagram. The obtained diagram is either parabolic or elliptic.*

A hyperbolic Coxeter diagram S is called a *Lannér diagram* if any proper subdiagram of S is elliptic. All Lannér diagrams were classified by Lannér in [Lan50]. They are listed in Table 2.3. These diagrams correspond (in the sense defined further) to compact hyperbolic Coxeter simplices. Nevertheless, the importance of such diagrams can already be appreciated.

Proposition 2.6. *Every hyperbolic diagram contains either a parabolic or a Lannér subdiagram.*

2.3 Hyperbolic polytopes

The *Minkowski space* $\mathbb{R}^{n,1}$ is the real vector space

$$\mathbb{R}^{n+1} = \{(x_0, x_1, \ldots, x_n) \mid x_i \in \mathbb{R},\ i = 0, \ldots, n\}$$

equipped with the indefinite scalar product

$$\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n.$$

Consider the two–sheeted hyperboloid

$$H = \{x \in \mathbb{R}^{n,1} \mid \langle x, x \rangle = -1\}.$$

The *hyperbolic space* is its upper half–sheet

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n,1} \mid \langle x, x \rangle = -1,\ x_0 > 0\}$$

with the induced metric. Even though the Minkowski scalar product is indefinite, it induces a Riemannian metric on the hyperbolic space $\mathbb{H}^n$. This metric turns out to have constant sectional curvature -1.

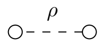
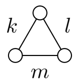
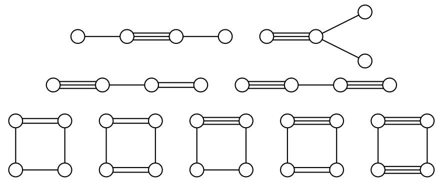
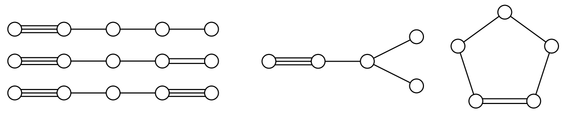
Order	Diagrams
2	 $\rho > 1$
3	 $\left(2 \leq k, l, m < \infty, \; \frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1\right)$
4	
5	

Table 2.3: Lannér diagrams

The central projection of $\mathbb{H}^n$ onto the plane $x_0 = 1$ through the origin produces an open ball. Its boundary is called the *ideal boundary* $\partial\mathbb{H}^n$ of the hyperbolic space $\mathbb{H}^n$. The points of $\partial\mathbb{H}^n$ correspond to the isotropic vectors

$$\{x \in \mathbb{R}^{n,1} \mid \langle x, x \rangle = 0,\ x_0 > 0\} \,/\, \mathbb{R}_{>0}\,.$$

The union $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial\mathbb{H}^n$ is called the *compactification* of $\mathbb{H}^n$.

Any vector $e \in \mathbb{R}^{n,1}$ with $\langle e, e \rangle = 1$ defines the associated *hyperbolic hyperplane*

$$H_e = \mathbb{H}^n \cap \{x \mid \langle e, x \rangle = 0\}$$

and the respective closed *hyperbolic half–space*

$$H_e^- = \mathbb{H}^n \cap \{x \mid \langle e, x \rangle \leqslant 0\}.$$

By $\overline{H_e}$ we denote the closure of H_e in $\overline{\mathbb{H}^n}$. If $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 1$ and $\langle e_1, e_2 \rangle \leqslant 0$ then the following holds:

(1) if $\langle e_1, e_2 \rangle > -1$, then the hyperplanes H_{e_1} and H_{e_2} intersect and the angle $\phi = \angle(H_{e_1}^-, H_{e_2}^-)$ can be found from the equation $\cos\phi = -\langle e_1, e_2 \rangle$;

(2) if $\langle e_1, e_2 \rangle = -1$, then the hyperplanes H_{e_1} and H_{e_2} do not intersect while their closures $\overline{H_{e_1}}$ and $\overline{H_{e_2}}$ share a unique point on the boundary $\partial\mathbb{H}^n$;

(3) if $\langle e_1, e_2 \rangle < -1$, then the closures $\overline{H_{e_1}}$ and $\overline{H_{e_2}}$ do not intersect, and the distance ρ between H_{e_1} and H_{e_2} measured along their unique common perpendicular can be found from the equation $\cosh\rho = -\langle e_1, e_2 \rangle$.

A *convex hyperbolic n–dimensional polyhedron* P is the intersection of finitely many closed half–spaces of $\mathbb{H}^n$. We also assume that the interior of P is non–empty. In the Klein model of the hyperbolic space $\mathbb{H}^n$ the closure $\overline{P} \subset \overline{\mathbb{H}^n}$ of a convex hyperbolic polyhedron $P \subset \mathbb{H}^n$ is the intersection of a convex Euclidean polytope with the unit ball centred at the origin (see [Vin93a]). So, we apply basically the usual Euclidean terms (e.g., faces and vertices) to hyperbolic polyhedra.

We say that a vertex v of $\overline{P} \subset \overline{\mathbb{H}^n}$ is a *finite vertex* if $v \in \mathbb{H}^n$ and an *ideal vertex* if $v \in \partial\mathbb{H}^n$. A hyperbolic polyhedron has a finite volume if and only if it coincides with the convex hull of its vertices. A finite volume hyperbolic polyhedron is compact if and only if all of its vertices are finite. In this case, the polyhedron is also called a *polytope*. If a finite volume hyperbolic polyhedron P has *only* ideal vertices, then P is called *ideal*.

2.4 Hyperbolic Coxeter polytopes

Let $P \subset \mathbb{H}^d$ be a Coxeter polytope with facets $f_1, \dots, f_n$. The Coxeter diagram $S(P)$ of the polytope P is a Coxeter diagram with vertices $v_1, \dots, v_n$. If the facets f_i and f_j intersect, then the weight of the edge $v_i v_j$ is equal to the cosine of the dihedral angle between the facets. If the facets f_i and f_j are parallel, then the weight of the edge $v_i v_j$ is equal to one. If the facets f_i and f_j diverge, then the weight of the edge $v_i v_j$ is equal to the hyperbolic cosine of the distance between f_i and f_j.

Now let us list the essential results on combinatorics of compact hyperbolic Coxeter polytopes. Let P be a polytope. By $\mathcal{F}(P)$ we denote the partially ordered set of its faces. Let S be a Coxeter diagram. By $\mathcal{F}(S)$ we denote the dual (i.e., anti-isomorphic to the original) partially ordered set of its elliptic subdiagrams.

Proposition 2.7 ([Vin85, Theorem 3.1])**.** *Let $P \subset \mathbb{H}^d$ be a compact hyperbolic Coxeter polytope. Partially ordered sets $\mathcal{F}(S(P))$ and $\mathcal{F}(P)$ are isomorphic.*

Thus, the combinatorics of a compact polytope can be easily read according to its Coxeter diagram. A set of facets has a non-empty intersection if and only if the subdiagram generated by the corresponding vertices is elliptic.

Consider a compact hyperbolic Coxeter polytope. The structure of its Coxeter diagram is restricted by the propositions below.

Proposition 2.8 ([Vin85, Proposition 3.2])**.** *Let $P \subset \mathbb{H}^d$ be a compact hyperbolic Coxeter polytope. The Coxeter diagram $S(P)$ contains no parabolic subdiagrams.*

Proposition 2.9 ([Vin85, Proposition 4.2])**.** *A Coxeter diagram S is a Coxeter diagram of a compact hyperbolic Coxeter polytope if and only if the diagram is hyperbolic, contains no parabolic subdiagrams, and there is a polytope $P \subset \mathbb{E}^d$ such that $\mathcal{F}(P)$ and $\mathcal{F}(S)$ are isomorphic.*

The following statement is an easy corollary of the propositions above.

Corollary 2.10. *A polytope $P \subset \mathbb{H}^d$ is a compact simplex if and only if $S(P)$ is a Lannér diagram.*

Finally, the best known general estimation on dimension of a compact hyperbolic Coxeter polytope is the following.

Theorem 2.11 ([Vin84, Theorem 1])**.** *There are no compact Coxeter polytopes in $\mathbb{H}^{\geqslant 30}$.*

2.5 Superhyperbolic diagrams

A Coxeter diagram is said to be *superhyperbolic* if its negative inertia index is greater than one. A *local determinant* of a diagram S on its subdiagram T is

$$\det(S, T) = \frac{\det(S)}{\det(S \setminus T)}.$$

Usually we will mark the vertices of the subdiagram T with $\vee$.

We denote by $p(\gamma)$ the product of the edge weights of a cycle γ. The following proposition is very useful for computing determinants.

Proposition 2.12 ([Vin84, Proposition 11])**.** *A determinant of a Coxeter diagram S is equal to the sum of the products*

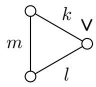
$$(-1)^k \cdot p(\gamma_1) \cdot \ldots \cdot p(\gamma_k)$$

over all sets $\{\gamma_1, \dots, \gamma_k\}$ of positive length disjoint cycles.

Proposition 2.13 ([Vin84, Proposition 13])**.** *If a Coxeter diagram S is generated by two disjoint subdiagrams S_1 and S_2 joined by a unique edge $v_1 v_2$ of weight w, then*

$$\det(S, \langle v_1, v_2 \rangle) = \det(S_1, v_1) \cdot \det(S_2, v_2) - w^2.$$

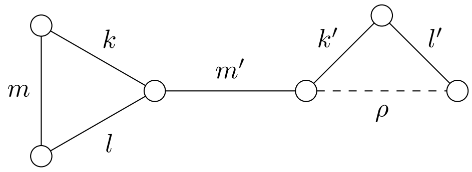
Proposition 2.14 ([Vin84, Table 2])**.**

$$\det\left(\right. $$  $$\left. \right) = -d(k,l,m),$$

where

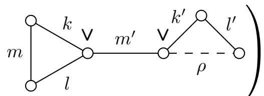
$$d(k,l,m) = \frac{\cos\left(\frac{\pi}{k}\right)^2 + \cos\left(\frac{\pi}{l}\right)^2 + 2\cos\left(\frac{\pi}{k}\right)\cos\left(\frac{\pi}{l}\right)\cos\left(\frac{\pi}{m}\right)}{\sin\left(\frac{\pi}{m}\right)^2} - 1.$$

Now let us use these propositions to test the diagram below for hyperbolicity.

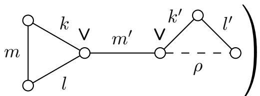
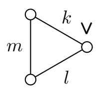
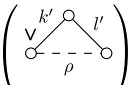


(2.1)

This diagram contains an elliptic subdiagram of order 4 and a Lannér subdiagram of order 2. Therefore, its signature is either $(4,1,1)$, or $(5,1,0)$, or $(4,2,0)$. Hence, the diagram is hyperbolic if and only if

$$\det\left(\right. $$  $$\left. \right) \leqslant 0. \tag{2.2}$$

But

$$\det\left(\right. $$  $$\left. \right) = \det\left(\right. $$  $$\left. \right) \cdot \det\left(\right. $$  $$\left. \right) - \cos\left(\frac{\pi}{m'}\right)^2 =$$

$$= d(k,l,m) \cdot \frac{\cos\left(\frac{\pi}{l'}\right)^2 + \cos\left(\frac{\pi}{k'}\right)^2 + \rho^2 + 2\rho\cos\left(\frac{\pi}{k'}\right)\cos\left(\frac{\pi}{l'}\right) - 1}{\sin\left(\frac{\pi}{l'}\right)^2} - \cos\left(\frac{\pi}{m'}\right)^2.$$

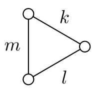
If $d(k,l,m) \neq 0$, then (2.2) is equivalent to the following:

$$\rho^2 + 2\rho\cos\left(\frac{\pi}{k'}\right)\cos\left(\frac{\pi}{l'}\right) + \cos\left(\frac{\pi}{l'}\right)^2 + \cos\left(\frac{\pi}{k'}\right)^2 - 1 - \frac{\sin\left(\frac{\pi}{l'}\right)^2\cos\left(\frac{\pi}{m'}\right)^2}{d(k,l,m)} \leqslant 0.$$

Consider the left part of this inequality as a quadratic function in ρ. One of the zeros of this function is not greater than 1. So there is a $\rho > 1$ satisfying the inequality if and only if for $\rho = 1$ the strict inequality holds, i.e.

$$D(k,l,m,k',l',m') = \left(\cos\left(\frac{\pi}{l'}\right) + \cos\left(\frac{\pi}{k'}\right)\right)^2 - \frac{\sin\left(\frac{\pi}{l'}\right)^2\cos\left(\frac{\pi}{m'}\right)^2}{d(k,l,m)} < 0. \tag{2.3}$$

This proves the following lemma.

Lemma 2.15. *Let*  *be a Lannér diagram. The Coxeter diagram (2.1) is superhyperbolic for every $\rho > 1$ if and only if*

$$D(k,l,m,k',l',m') \geqslant 0,$$

where $D(k,l,m,k',l',m')$ is as in (2.3).

Remark 2.16. Direct calculations show that if $d(k,l,m) > 0$, then the function D is increasing in k, l, m, k', l', and decreasing in m'.

2.6 Hyperbolic right-angled polyhedra

The combinatorics of hyperbolic acute-angled polyhedra in $\mathbb{H}^3$ was well studied by Andreev (see [And70a, And70b]). The following theorem is a special case of Andreev's theorem.

Theorem 2.17. *Let $\mathcal{P}$ be a combinatorial 3-polytope. There exists a finite volume right-angled hyperbolic 3-polyhedron $P \subset \overline{\mathbb{H}^3}$ that realises $\mathcal{P}$ if and only if:*

(1) $\mathcal{P}$ is neither a tetrahedron, nor a triangular prism;

(2) every vertex of $\mathcal{P}$ belongs to at most four faces;

(3) if f, f', and f'' are faces of $\mathcal{P}$, and $e' = f \cap f'$, $e'' = f \cap f''$ are non-intersecting edges, then f' and f'' do not intersect each other;

(4) there are no faces f_1, f_2, f_3, f_4 such that $e_i := f_i \cap f_{i+1}$ (indices $\bmod 4$) are pairwise non-intersecting edges of $\mathcal{P}$.

In a right-angled polyhedron $P \subset \overline{\mathbb{H}^3}$, a vertex v lies in $\mathbb{H}^3$ (i.e. is finite) if and only if it belongs to exactly three faces of P. If a vertex v is contained in four faces of P, then $v \in \partial\mathbb{H}^3$.

2.7 Combinatorics of Euclidean polytopes

Let us consider an n-dimensional convex Euclidean polytope P. We say that P is *simplicial* if every facet of P is a simplex. We say that P is *simple* if every vertex of P is the intersection of exactly n facets. The polar dual of a simplicial polytope is a simple one, and vice versa.

Let f_k denote the number of k-dimensional faces of P (here and below we assume that $f_{-1} = f_n = 1$). With each n-polytope we associate the n-dimensional vector $f = (f_0, f_1, \ldots, f_{n-1})$, which is called the *$f$-vector* of the polytope. Euler's theorem states that

$$\sum_{i=0}^{n-1} f_i = 1 - (-1)^n$$

and, therefore, that there is a $(n-1)$-dimensional (affine) subspace that contains the f-vector of every n-dimensional polytope. It turns out that for simplicial polytopes there are more linear restrictions on their f-vectors.

Theorem 2.18 (Dehn–Sommerville equations)**.** *For every n-dimensional simplicial convex Euclidean polytope the following equations hold:*

$$\sum_{j=k}^{n-1} (-1)^j \binom{j+1}{k+1} f_j = (-1)^{n-1} f_k,$$

where $k = -1, 0, \ldots, n-2$.

Corollary 2.19. *For every 7-dimensional simplicial convex Euclidean polytope the following equalities hold:*

$$\begin{aligned}
f_3 &= 5f_2 - 15f_1 + 35f_0 - 70,\\
f_4 &= 9f_2 - 34f_1 + 84f_0 - 168,\\
f_5 &= 7f_2 - 28f_1 + 70f_0 - 140,\\
f_6 &= 2f_2 - 8f_1 + 20f_0 - 40.
\end{aligned}$$

Corollary 2.20. *For every 13-dimensional simplicial convex Euclidean polytope the following equalities hold:*

$$\begin{aligned}
f_6 &= 8f_5 - 36f_4 + 120f_3 - 330f_2 + 792f_1 - 1716f_0 + 3432,\\
f_7 &= 27f_5 - 159f_4 + 585f_3 - 1683f_2 + 4125f_1 - 9009f_0 + 18018,\\
f_8 &= 50f_5 - 325f_4 + 1252f_3 - 3685f_2 + 9130f_1 - 20\,020f_0 + 40\,040,\\
f_9 &= 55f_5 - 374f_4 + 1474f_3 - 4389f_2 + 10\,934f_1 - 24\,024f_0 + 48\,048,\\
f_{10} &= 36f_5 - 250f_4 + 996f_3 - 2982f_2 + 7448f_1 - 16\,380f_0 + 32\,760,\\
f_{11} &= 13f_5 - 91f_4 + 364f_3 - 1092f_2 + 2730f_1 - 6006f_0 + 12\,012,\\
f_{12} &= 2f_5 - 14f_4 + 56f_3 - 168f_2 + 420f_1 - 924f_0 + 1848.
\end{aligned}$$

2.8 Combinatorics of ideal right-angled hyperbolic 3-polyhedra

Let P be an ideal hyperbolic right-angled 3-polyhedron. Let V be the number of vertices, E the number of edges, and F the number of faces of P. The Euler characteristic of P equals $V - E + F = 2$. Every ideal vertex of a finite volume right-angled hyperbolic 3-polyhedron is contained in exactly four edges which implies $4V = 2E$, and hence $F = V + 2$. Let p_k denote the number of k-gonal faces of P. Then the previous equalities provide

$$\sum_{k\geqslant 3} p_k = F, \quad \sum_{k\geqslant 3} kp_k = 4V, \quad \text{and} \quad p_3 = 8 + \sum_{k\geqslant 5}(k-4)p_k.$$

We say that two vertices of P are *adjacent* if they are connected by an edge. Two vertices are *quasi-adjacent* if they belong to the same face but are not adjacent.

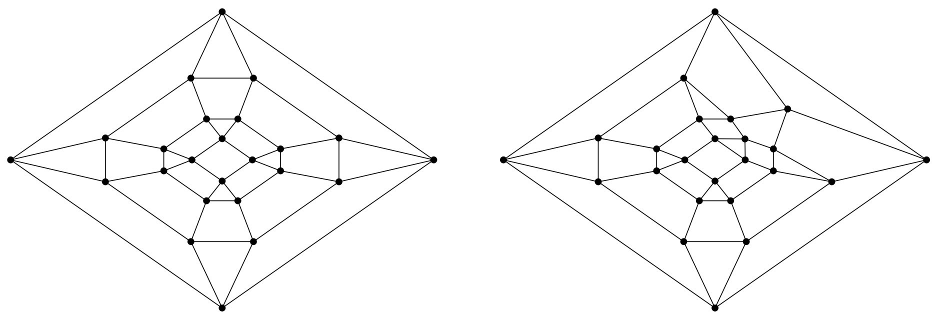


Figure 2.1: Ideal right-angled polyhedra with $V = 24$ vertices.

Example 2.21. Two ideal right-angled polyhedra with 24 vertices from [EV20a] are shown in Figure 2.1. Each vertex of those polyhedra belongs to exactly one triangular face and three quadrilateral faces. Thus, each vertex of those polyhedra has exactly three quasi-adjacent vertices.

Lemma 2.22. *Let P be an ideal right-angled hyperbolic 3-polyhedron with $V > 24$ vertices. Then there is a vertex that has at least 4 quasi-adjacent ones.*

Proof. Let $q(v)$ be the number of vertices that are quasi-adjacent to v. Then the average number of quasi-adjacent vertices in P equals

$$\begin{aligned}
\frac{\sum_v q(v)}{V} &= \frac{1}{V}\sum_{k\geqslant 3} k(k-3)p_k = \frac{1}{V}\sum_{k\geqslant 3} k^2 p_k - \frac{3}{V}\sum_{k\geqslant 3} kp_k = \\
&= \frac{1}{V}\sum_{k\geqslant 3} k^2 p_k - 12 = \frac{1}{V}\left[3^2\, p_3 + 4^2\left(F - p_3 - \sum_{k\geqslant 5} p_k\right) + \sum_{k\geqslant 5} k^2 p_k\right] - 12, \\
&= \frac{1}{V}\left[3^2\left(8 + \sum_{k\geqslant 5}(k-4)p_k\right) + 4^2\left(F - p_3 - \sum_{k\geqslant 5} p_k\right) + \sum_{k\geqslant 5} k^2 p_k\right] - 12 = \\
&= \frac{1}{V}\left[3^2\left(8 + \sum_{k\geqslant 5}(k-4)p_k\right) + 4^2\left(V - 6 - \sum_{k\geqslant 5}(k-3)p_k\right) + \sum_{k\geqslant 5} k^2 p_k\right] - 12 = \\
&= 4 - \frac{24}{V} + \frac{1}{V}\sum_{k\geqslant 5}(k^2 - 7k + 12)p_k \geqslant 4 - \frac{24}{V} > 3.
\end{aligned}$$

Therefore, there is a vertex in P that has at least 4 quasi-adjacent vertices. ☐

Remark 2.23. Each vertex of a k-gonal face has at least $k-3$ quasi-adjacent ones. So if a polytope has a k-gonal face for $k \geqslant 7$ then it has a vertex with at least 4 quasi-adjacent ones.

We say that a face f and a vertex v are *incident* if v belongs to f. A face f and a vertex v are *quasi-incident* if they are not incident, but v has an incident face f' such that f' shares an edge with f.

Proposition 2.24. *Let P be an ideal right-angled hyperbolic 3-polyhedron with $V > 72$ vertices and with only triangular and quadrilateral faces. Then there is a vertex without incident and quasi-incident triangular faces, see Figure 2.2.*

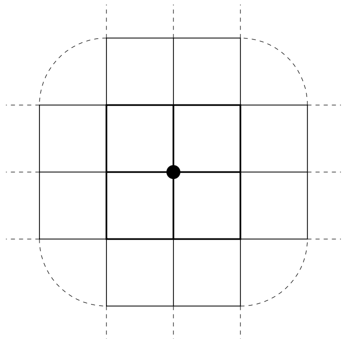


Figure 2.2: A vertex without incident and quasi-incident triangular faces.

Proof. Let F denote the number of faces of P. Because there are only triangular and quadrilateral faces, the set of faces of P consists of 8 triangles and $F - 8$ quadrilaterals. Every triangular face

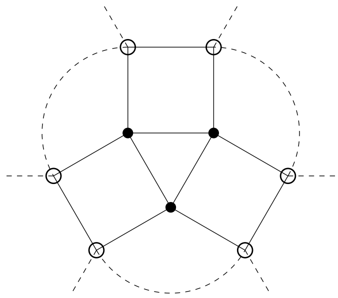


Figure 2.3: Incident and quasi-incident vertices of a 3-gonal face.

is incident or quasi-incident to at most 9 vertices (see Figure 2.3). Therefore, at most 72 vertices of P can be incident or quasi-incident to triangular faces. □

2.9 Combinatorics of compact right-angled hyperbolic 3-polytopes

Let P be a compact right-angled hyperbolic 3-polytope. Let V denote the number of vertices, E the number of edges, and F the number of faces. The Euler characteristic of P equals $V - E + F = 2$. Every vertex of a compact right-angled hyperbolic 3-polytope is incident to three edges, which implies $3V = 2E$ and $F = \frac{1}{2}V + 2$. Let p_k denote the number of k-gonal faces of P. By Theorem 2.17 $p_3 = 0$ and $p_4 = 0$, and the previous equalities imply

$$\sum_{k \geqslant 5} p_k = F, \qquad \sum_{k \geqslant 5} k p_k = 3V, \quad \text{and} \quad p_5 = 12 + \sum_{k \geqslant 7} (k-6) p_k.$$

An edge e and a vertex v are *incident* if v is one of the two vertices that e connects. We say that an edge e and a vertex v are *quasi-incident* if they are not incident, but at least one vertex of e belongs to the same face as v.

Since each vertex of a compact right-angled hyperbolic polyhedron P is trivalent, we have four faces f_1, f_2, f_3, and f_4 arranged around each edge e of P as shown in Figure 2.4. If f_i is k_i-gonal, then number of vertices quasi-incident to e is equal to $\sum_{i=1}^{4} k_i - 10$ (note that Andreev's theorem implies that $f_2 \cap f_4 = \varnothing$).

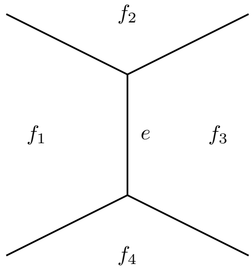


Figure 2.4: An edge e and faces around.

Example 2.25. Consider the polyhedron presented in Figure 2.5. It has 80 vertices and it is known

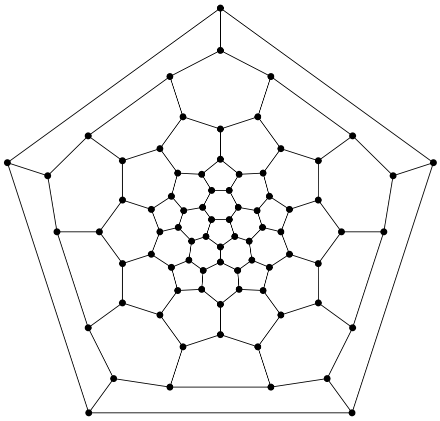


Figure 2.5: Fullerene C80.

as fullerene C80 is the structural chemistry. Each edge of C80 is quasi-incident to 13 vertices. Indeed, every edge of C80 has one pentagon and three hexagons around itself.

Lemma 2.26. *Let P be a compact right-angled hyperbolic polytope with $V > 80$ vertices. Then there is an edge with at least 14 quasi-incident vertices.*

Proof. Let $q(e)$ be the number of vertices that are quasi-incident to an edge e. Then the average number of quasi-incident vertices in P equals

$$\frac{\sum_e q(e)}{E} = \frac{1}{E}\left[\sum_{k\geqslant 5} k(k-2)p_k + \sum_{k\geqslant 5} k(k-3)p_k\right] = \frac{2}{E}\sum_{k\geqslant 5} k^2 p_k - \frac{5}{E}\sum_{k\geqslant 5} kp_k$$

$$= \frac{2}{E}\sum_{k\geqslant 5} k^2 p_k - 10 = \frac{2}{E}\left[5^2 p_5 + 6^2 p_6 + \sum_{k\geqslant 7} k^2 p_k\right] - 10$$

$$= \frac{2}{E}\left[5^2\left(12 + \sum_{k\geqslant 7}(k-6)p_k\right) + 6^2\left(F - p_5 - \sum_{k\geqslant 7} p_k\right) + \sum_{k\geqslant 7} k^2 p_k\right] - 10$$

$$= \frac{2}{E}\left[5^2\left(12 + \sum_{k\geqslant 7}(k-6)p_k\right) + 6^2\left(\frac{E}{3} - 10 - \sum_{k\geqslant 7}(k-5)p_k\right) + \sum_{k\geqslant 7} k^2 p_k\right] - 10$$

$$= 14 - \frac{120}{E} + \frac{2}{E}\sum_{k\geqslant 7}(k^2 - 11k + 30)p_k \geqslant 14 - \frac{120}{E} = 14 - \frac{80}{V} > 13.$$

Therefore, there exists an edge with at least 14 quasi-incident vertices. □

Corollary 2.27. *Suppose that P is a compact right-angled hyperbolic 3-polytope with $V > 80$ vertices. Then there is an edge with k_i-gonal faces around, $i = 1, \ldots, 4$, such that $\sum_{i=1}^{4} k_i \geqslant 24$.*

2.10 Combinatorics of finite volume right-angled hyperbolic 3-polyhedra

Let P be a right-angled hyperbolic 3-polyhedron with V_F finite and V_∞ ideal vertices. Denote the number of its edges by E, and the number of its faces by F. The Euler characteristic of P equals

$$V_F + V_\infty - E + F = 2.$$

Since every ideal vertex is incident to four edges and each finite vertex is incident to three edges, we get $3V_F + 4V_\infty = 2E$. Hence $F = \frac{1}{2}V_F + V_\infty + 2$. We say that two faces are *neighbours* if they have a common vertex.

Lemma 2.28. *Let P be a finite volume right-angled hyperbolic 3-polyhedron with V_F finite and V_∞ ideal vertices.*

(1) Suppose $V_F + V_\infty > 15$ and $V_\infty \geqslant 1$, or $V_F + V_\infty \geqslant 15$ and $V_\infty > 1$. Then there is a face $f \in P$ with at least 6 neighbours.

(2) Let $V_F + V_\infty > 17$ and $V_\infty \geqslant 3$. If there is no face with $\geqslant 7$ neighbours, then there are at least 7 faces such that each of them has 6 neighbours.

(3) If $V_\infty \geqslant 6$ and there is a face $f \in P$ with at most 5 neighbours, then there is a face $f' \in P$ with at least 7 neighbours.

Proof. Suppose that P has F faces. For a face $f_i \in P$, $i = 1, \ldots, F$, denote by V_F^i the number of finite vertices in f_i and by V_∞^i the number of ideal vertices in f_i. Then the average number A of neighbouring faces in P is equal to

$$A = \frac{1}{F}\sum_i (2V_\infty^i + V_F^i) = \frac{1}{F}(8V_\infty + 3V_F) = \frac{8V_\infty + 3V_F}{V_\infty + \frac{1}{2}V_F + 2}. \tag{2.4}$$

(1) Our aim is to show that

$$\frac{8V_\infty + 3V_F}{V_\infty + \frac{1}{2}V_F + 2} > 5,$$

which is equivalent to

$$8V_\infty + 3V_F > 5V_\infty + \frac{5}{2}V_F + 10$$

and

$$3V_\infty + \frac{1}{2}V_F > 10.$$

Using $V_F + V_\infty > 15$, we obtain

$$3V_\infty + \frac{1}{2}V_F > 3V_\infty + \frac{15}{2} - \frac{1}{2}V_\infty = \frac{5}{2}V_\infty + \frac{15}{2} \geqslant 10$$

if $V_\infty \geqslant 1$. Thus, there is a face $f \in P$ with at least 6 neigbouring faces. Analogously, using $V_F + V_\infty \geqslant 15$, we obtain

$$3V_\infty + \frac{1}{2}V_F \geqslant 3V_\infty + \frac{15}{2} - \frac{1}{2}V_\infty = \frac{5}{2}V_\infty + \frac{15}{2} > 10$$

if $V_\infty > 1$. Thus, there is a face $f \in P$ with at least 6 neigbouring faces.

(2) Let us use the formula (2.4) for the average number A of neighbours:

$$A = \frac{8V_\infty + 3V_F}{V_\infty + \frac{1}{2}V_F + 2} = \frac{6(V_\infty + \frac{1}{2}V_F + 2) + 2V_\infty - 12}{V_\infty + \frac{1}{2}V_F + 2} \geqslant$$

$$\geqslant \frac{6(V_\infty + \frac{1}{2}V_F + 2) - 6}{V_\infty + \frac{1}{2}V_F + 2} = 6 - \frac{6}{V_\infty + \frac{1}{2}V_F + 2},$$

where we used $2V_\infty - 6 \geqslant 0$. Since $V_\infty + V_F > 17$ and $V_\infty \geqslant 3$, we have

$$V_\infty + \frac{1}{2}V_F + 2 \geqslant 11 + \frac{1}{2}V_\infty \geqslant \frac{25}{2}.$$

Hence the average number of neighbours satisfies the following inequality

$$A \geqslant 6 - \frac{12}{25} = \frac{138}{25}.$$

Since $V_\infty + V_F > 17$ and $V_\infty \geqslant 3$, a polyhedron P has

$$F = \frac{1}{2}V_F + V_\infty + 2 \geqslant 11 + \frac{1}{2}V_\infty > 12$$

faces. But F is an integer number, so $F \geqslant 13$. Assume that P has $k \leqslant 6$ faces with 6 neighbours. Hence P has $F - k$ faces with at most 5 neighbours, and an average number A of neighbours in P satisfies the following inequality

$$A \leqslant \frac{6k + 5(F-k)}{F} = 5 + \frac{k}{F} \leqslant 5 + \frac{6}{13} = \frac{71}{13}.$$

Since $\frac{71}{13} < \frac{138}{25}$, we get a contradiction. Hence P has at least 7 faces such that each of them has 6 neighbouring faces.

(3) By the formula (2.4), using $V_\infty \geqslant 6$, we get the following inequality for the average number of neighbours:

$$A = \frac{8V_\infty + 3V_F}{V_\infty + \frac{1}{2}V_F + 2} = \frac{6(V_\infty + \frac{1}{2}V_F + 2) + 2V_\infty - 12}{V_\infty + \frac{1}{2}V_F + 2} \geqslant 6.$$

Since $A \geqslant 6$ and f has at most 5 neighbours, there is a face f' of P with at least 7 neighbours. $\square$

Chapter 3

Proofs

3.1 Proof of Theorems 1.1 and 1.2

Let Σ_1 and Σ_2 be sets of Coxeter diagram. By $\Sigma_1 \times_k \Sigma_2$ we denote the set of all Coxeter diagrams S generated by subdiagrams $S_1 \in \Sigma_1$ and $S_2 \in \Sigma_2$ such that intersection $S_1 \cap S_2$ consists of k vertices and every Lannér or parabolic subdiagram is contained in either S_1 or S_2.

Denote by $\mathcal{L}_k$ the set of all Lannér diagrams of order k and by Δ_k the standard $(k-1)$-dimensional simplex. Consider a compact hyperbolic Coxeter polytope P. Suppose that $\mathcal{F}(P)$ and $\mathcal{F}(\Delta_{k_1} \times \cdots \times \Delta_{k_n})$ are isomorphic. Every face of $\Delta_{k_1} \times \cdots \times \Delta_{k_n}$ is equal to $f_1 \times \cdots \times f_n$ for some faces f_i of Δ_i. Therefore, the facets of $\Delta_{k_1} \times \cdots \times \Delta_{k_n}$ are equal to

$$f_j^i = \Delta_{k_1} \times \cdots \times \Delta_{k_{i-1}} \times f_j \times \Delta_{k_{i+1}} \times \cdots \times \Delta_{k_n}, \quad \text{where } f_j \text{ is a facet of } \Delta_{k_i}.$$

Let F be a set of the facets. The intersection $\bigcap_{f \in F} f$ is empty if and only if $\{f_1^i, \ldots, f_{k_i}^i\} \subseteq F$ for some $1 \leqslant i \leqslant n$. According to Proposition 2.6 and Proposition 2.9, $S(P) \in \mathcal{L}_{k_1} \times_0 \cdots \times_0 \mathcal{L}_{k_n}$.

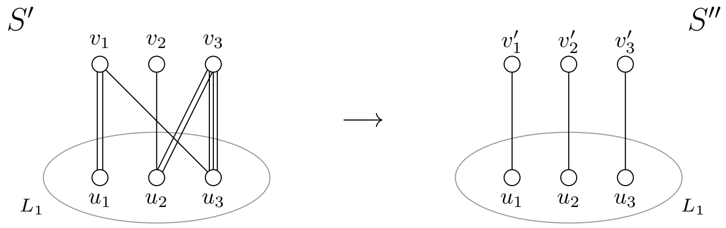
Without loss of generality, $k_1 \geqslant \cdots \geqslant k_n$. If $k_1 = \cdots = k_n = 2$, then P is a n-dimensional cube. If $k_n \neq 2$, then the diagram $S(P)$ contains no dashed edges. It is known that every such polytope is a product of at most two simplices (see [FT08, Theorem A]). Thus, Theorem 1.2 is a corollary of Theorem 1.1. For the reader's convenience we present its statement again.

Theorem 1.1 ([Ale22, Theorem A])**.** *Let $n \geqslant 4$ and $2 \neq k_1 \geqslant \cdots \geqslant k_n = 2$. Every diagram contained in the set $\mathcal{L}_{k_1} \times_0 \cdots \times_0 \mathcal{L}_{k_n}$ is superhyperbolic.*

The plan of the proof is as follows. At first we show that if at least one Lannér subdiagram has order $\geqslant 4$ then the set under consideration is empty (Subsection 3.1.1). After that we prove that if the diagram contains at least two Lannér subdiagrams of order 3 then it is superhyperbolic (Subsection 3.1.2). Finally, we deal with the product of a Lannér diagram of order 3 with several other Lannér diagrams of order 2 (Subsection 3.1.3). In order to do this we prove that the product of a Lannér diagram of order 3 and four Lannér diagrams of order 2 is superhyperbolic. The rest of the subsection is devoted to proving that the product of a Lannér diagram of order 3 and three Lannér diagrams of order 2 is superhyperbolic.

3.1.1 Case $k_1 \geqslant 4$

Let $S = \langle L_1, \ldots, L_n \rangle$ be a Coxeter diagram generated by disjoint Lannér diagrams $L_1, \ldots, L_n$ of orders $k_1 \geqslant \cdots \geqslant k_n$. Assume that $S \in \mathcal{L}_{k_1} \times_0 \cdots \times_0 \mathcal{L}_{k_n}$. For any $i = 2, \ldots, n$ the diagram L_1 has at least one adjacent vertex $v_i \in L_i$ (due to Proposition 2.1). The diagram $S' = \langle L_1, v_2, \ldots, v_n \rangle$ contains exactly one Lannér subdiagram L_1. In other words, a subdiagram of the diagram S' is hyperbolic if and only if it contains L_1. Denote by u_i an arbitrary vertex of L_1 that is attached to v_i. Let us construct a new diagram $S'' = \langle L_1, v_2', \ldots, v_n' \rangle$, where v_i' is adjoint only to u_i by a simple edge.



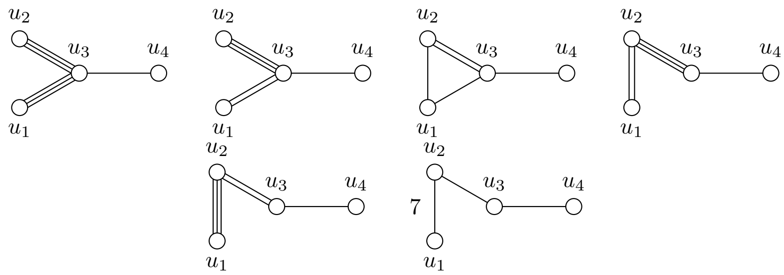
Corollary 2.4 implies that a subdiagram of S'' is elliptic if and only if it contains the subdiagram L_1. Therefore, L_1 is the only Lannér subdiagram of S''.

Consider a Lannér diagram L of order 4 or 5. Considering each of the finite number of such diagrams, it can be verified that if each of three vertices is attached by a simple edge to L (like in S''), then it contains either a parabolic subdiagram, or a Lannér subdiagram other than L. The same holds for S'' and therefore for S'. This disproves the assumption.

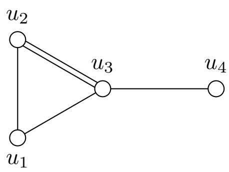
We proved that the sets $\mathcal{L}_4 \times_0 \{\circ\} \times_0 \{\circ\} \times_0 \{\circ\}$ and $\mathcal{L}_5 \times_0 \{\circ\} \times_0 \{\circ\} \times_0 \{\circ\}$ are empty ($\circ$ is a Coxeter diagram consisting of one vertex). In this case, we say that no Lannér diagram of order 4 or 5 can be *expanded* with three vertices without forming a new Lannér or parabolic subdiagram.

3.1.2 Case $k_1 = k_2 = 3$

Consider the Lannér subdiagrams $L_1 = \langle u_1, u_2, u_3 \rangle$ and $L_2 = \langle u_4, u_5, u_6 \rangle$ of order 3. It is shown in [Tum07, Lemma 4.10] that L_1 and L_2 are joined by a unique simple edge and if $|\det(L_1, u_3)| \leqslant |\det(L_2, u_4)|$, then without loss of generality the subdiagram $\langle L_1, u_4 \rangle$ is one of the following.



Applying the argument from Subsection 3.1.1, we get that the diagram L_1 can be expanded with two vertices. So the subdiagram $\langle L_1, u_4 \rangle$ is the following.



So, $|\det(L_1, u_3)| = \frac{\sqrt{2}}{3}$. According to Proposition 2.13,

$$\det\big(\langle L_1, L_2 \rangle, \langle u_3, u_4 \rangle\big) = \det(L_1, u_3) \cdot \det(L_2, u_4) - \frac{1}{4}.$$

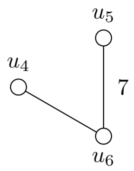
The determinants $\det\big(\langle L_1, L_2 \rangle, \langle u_3, u_4 \rangle\big)$, $\det(L_1, u_3)$, and $\det(L_2, u_4)$ are not positive. Therefore,

$$0 \geqslant \det\big(\langle L_1, L_2 \rangle, \langle u_3, u_4 \rangle\big) = |\det(L_2, u_4)| \cdot \frac{\sqrt{2}}{3} - \frac{1}{4}$$

and

$$|\det(L_2, u_4)| \leqslant \frac{3}{4\sqrt{2}}.$$

Note that the multiplicity of the edges u_4u_5 and u_4u_6 does not exceed one. There is the only Lannér diagram of order 3 with such properties, which is shown below.



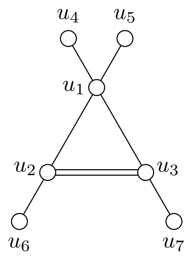
This diagram is not appropriate since it cannot be expanded with three vertices.

3.1.3 Case $k_2 = 2$

A Lannér diagram of order 3 cannot be expanded with five vertices. Therefore, $n \leqslant 5$. Let us denote by $[u, v]$ the multiplicity of the edge connecting vertices u and v.

Lemma 3.1. *Under the conditions described above, $n \leqslant 4$.*

Proof. Suppose that $n = 5$. Denote the Lannér subdiagrams by $L_1 = \langle u_1, u_2, u_3 \rangle$, $L_2 = \langle u_4, u_8 \rangle$, $L_3 = \langle u_5, u_9 \rangle$, $L_4 = \langle u_6, u_{10} \rangle$, and $L_5 = \langle u_7, u_{11} \rangle$. Without loss of generality, the vertices u_4, u_5, u_6, and u_7 are joined to the subdiagram L_1. The only subdiagram $\langle L_1, u_4, u_5, u_6, u_7 \rangle$ that satisfies these properties is shown below.



It is easy to check that

$$[u_6, u_4] = [u_6, u_5] = [u_6, u_8] = [u_6, u_9] =$$
$$[u_7, u_4] = [u_7, u_5] = [u_7, u_8] = [u_7, u_9] = 0.$$

This implies that the vertices u_{10} and u_{11} are joined to the subdiagrams $\langle u_4, u_8 \rangle$ and $\langle u_5, u_9 \rangle$. There are two cases:

1. Let $[u_{10}, u_{11}] \geqslant 1$. Without loss of generality, we may assume that

   $$[u_{10}, u_8] = [u_{10}, u_5] = [u_{11}, u_4] = [u_{11}, u_9] \geqslant 1$$

   and

   $$[u_{11}, u_8] = [u_{11}, u_5] = [u_{10}, u_4] = [u_{10}, u_9] = 0.$$

   Then

   $$[u_4, u_5] = [u_4, u_9] = [u_8, u_5] = [u_8, u_9] = 0$$

   and the subdiagram $\langle L_2, L_3 \rangle$ is not connected.

2. Let $[u_{10}, u_{11}] = 0$. Then, without loss of generality, we may assume that $[u_6, u_{11}] = 1$. In this case the subdiagram L_5 can be joined with L_2 and L_3 only if

   $$[u_{11}, u_8] = [u_{11}, u_9] \geqslant 1.$$

   Then

   $$[u_4, u_5] = [u_4, u_9] = [u_8, u_5] = [u_8, u_9] = 0$$

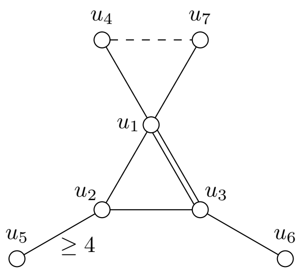
   and the subdiagram $\langle L_2, L_3 \rangle$ is not connected.

□

Thus, only the products of a triangle and a 3-dimensional cube left. Denote the Lannér subdiagrams by $L_1 = \langle u_1, u_2, u_3\rangle$, $L_2 = \langle u_4, u_7\rangle$, $L_3 = \langle u_5, u_8\rangle$, and $L_4 = \langle u_6, u_9\rangle$. We suppose that the subdiagrams $\langle L_1, u_4\rangle$, $\langle L_1, u_5\rangle$, and $\langle L_1, u_6\rangle$ are connected. If the subdiagram $\langle L_1, u_4, u_5, u_6\rangle$ contains the only Lannér subdiagram, then all edges of the subdiagram L_1 have a positive multiplicity. This means that any vertex of the subdiagrams L_2, L_3, and L_4 is joined to L_1 by at most one edge. Denote the multiplicity of such an edge by $[u, L_1]$. If $[u_7, L_1] \geqslant 1$ and $[u_8, L_1] \geqslant 1$, then L_2 and L_3 are not connected. Thus, without loss of generality, $[u_8, L_1] = [u_9, L_1] = 0$, $[u_4, L_1] \geqslant [u_7, L_1]$, and $[u_5, L_1] \geqslant [u_6, L_1] = 1$.

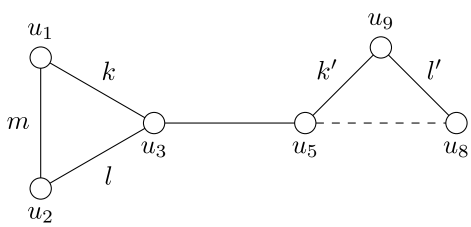
Lemma 3.2. *If* $[u_5, L_1] \geqslant 2$, *then* $[u_7, L_1] = 0$.

Proof. Assume that $[u_7, L_1] \geqslant 1$. The only possible subdiagram $\langle L_1, L_2, u_5, u_6\rangle$ is shown below.



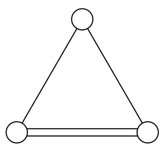
Then $[u_5, u_4] = [u_5, u_7] = [u_8, u_4] = [u_8, u_7] = 0$ and the subdiagrams L_2 and L_3 are not connected. □

We may suppose that $[u_5, L_1] = 1$ since otherwise we can swap L_2 and L_3. The vertex u_8 is joined to L_4 or the vertex u_9 is joined to L_3. Without loss of generality, u_9 is joined to L_3. The only possible diagram $\langle L_1, L_3, u_9\rangle$ is shown below, $k' \geqslant 3$ or $l' \geqslant 3$.

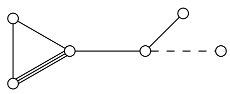
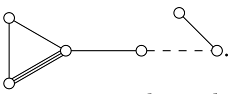


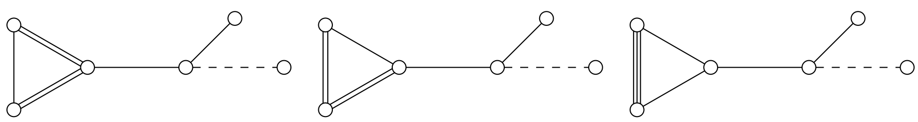
(3.1)

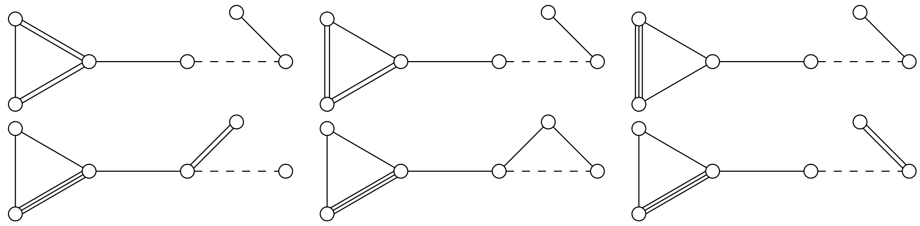
Lemma 3.3. *The diagram* L_1 *is equal to the following diagram.*



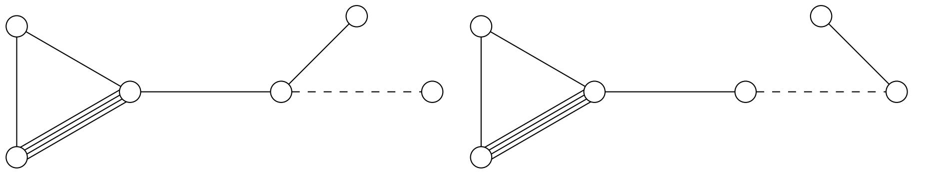
(3.2)

Proof. Either the diagram L_1 is equal to (3.2), or $\langle L_1, L_3, u_9\rangle$ is equal to  or . Indeed, assume the statement is false. The subdiagram $S = \langle L_1, L_3, u_9\rangle$ is not superhyperbolic and equal to (3.1). Let us decrease k, l, m, k' and l', and denote the obtained diagram by S'. Lemma 2.15 and monotonicity of the function D imply that S' is not superhyperbolic and Corollary 2.5 implies that S' does not contain parabolic subdiagrams. But by decreasing the multiplicities, one can obtain from S either one of the following superhyperbolic (by Lemma 2.15) diagrams

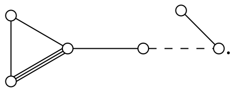


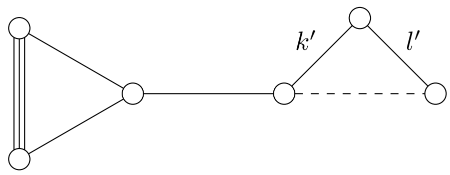


or one of the following diagrams containing a parabolic subdiagram.

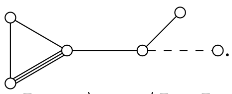


Therefore, the assumption is false.

Let $\langle L_1, L_3, u_9\rangle$ be equal to . The subdiagram $\langle L_1, L_4, u_8\rangle$ is equal to the following diagram.



$$k' \geqslant 3 \quad \text{or} \quad l' \geqslant 3 \tag{3.3}$$

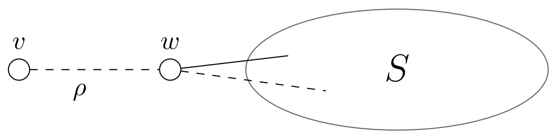
Let $\langle L_1, L_3, u_9\rangle$ be equal to . Since the subdiagram $\langle L_2, L_4\rangle$ is connected, then $[u_7, L_1] = 0$ and either $\langle L_1, L_2, u_9\rangle$ or $\langle L_1, L_4, u_7\rangle$ is equal to (3.3). To complete the proof, it remains to note that the diagram (3.3) is superhyperbolic by the argument from the beginning of the proof. $\square$

The next several pages will be devoted to proving that some diagrams are superhyperbolic.

Lemma 3.4. *Let S be a diagram that contains a hyperbolic subdiagram and let $v \notin S$ be a vertex that is joined with the only vertex $w \notin S$ by a dotted edge. If the inequality*

$$\det\big(\langle w, S\rangle\big) - \det(S) > 0$$

holds, then the diagram $\langle v, w, S\rangle$ is superhyperbolic.



Proof. Let us choose arbitrary labels on the dotted edges. Denote by ρ the label on the dotted edge between v and w. Direct calculation provides

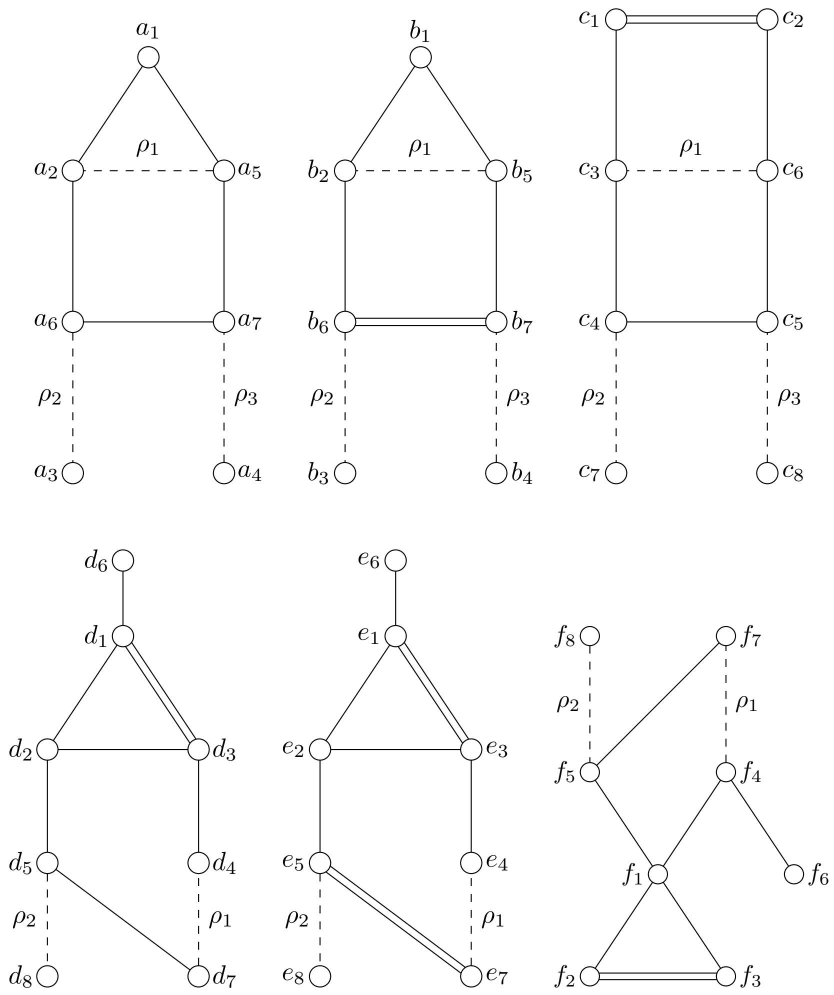
$$\det(\langle v, w, S\rangle) = \det(\langle w, S\rangle) - \rho^2 \det(S).$$

Suppose that the diagram $\langle v, w, S\rangle$ is hyperbolic. If $\det(S) < 0$, then

$$\rho \leqslant \sqrt{\frac{\det(\langle w, S\rangle)}{\det(S)}} = \sqrt{1 + \frac{\det(\langle w, S\rangle) - \det(S)}{\det(S)}} \leqslant 1.$$

We get $\det(S) = 0$ and $\det\big(\langle w, S\rangle\big) > 0$. Therefore, the diagram $\langle w, S\rangle$ is superhyperbolic. $\square$

Corollary 3.5. *The diagrams below are superhyperbolic for any $\rho_1, \rho_2, \rho_3 > 1$.*



Proof. For $\rho_1, \rho_2, \rho_3 > 1$ we have

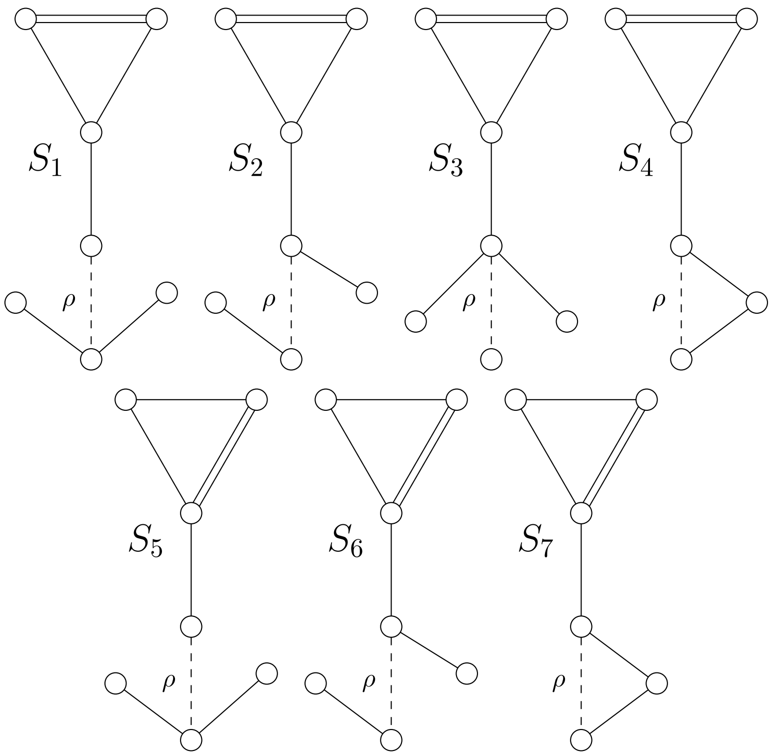
$$\begin{aligned}
\det(\langle a_7, A\rangle) - \det(A) &= \frac{1}{16}\big(3\rho_2^2 + 4\rho_1^2 - 2\rho_1 - 5\big) > 0,\\
\det(\langle b_7, B\rangle) - \det(B) &= \frac{1}{16}\big(3\rho_2^2 + 8\rho_1^2 - 4(\sqrt{2}-1)\rho_1 - 6 - \sqrt{2}\big) > 0,\\
\det(\langle c_5, C\rangle) - \det(C) &= \frac{1}{64}\big(4\rho_2^2 + 8\rho_1^2 - 4(2-\sqrt{2})\rho_1 - 2\sqrt{2} - 3\big) > 0,\\
\det(\langle d_5, D\rangle) - \det(D) &= \frac{1}{32}\big(2\rho_1^2 - (3+2\sqrt{2})\rho_1 + 2\sqrt{2} + 2\big) > 0,\\
\det(\langle e_5, E\rangle) - \det(E) &= \frac{1}{64}\big(4\rho_1^2 - 2(4+3\sqrt{2})\rho_1 + 8\sqrt{2} + 9\big) > 0,\\
\det(\langle f_5, F\rangle) - \det(F) &= \frac{1}{64}\big(8\rho_1^2 - 8\rho_1 + 3\sqrt{2} - 4\big) > 0,
\end{aligned}$$

where

$$\begin{aligned}
A &= \langle a_1, a_2, a_3, a_5, a_6\rangle, & D &= \langle d_1, d_2, d_3, d_4, d_6, d_7\rangle,\\
B &= \langle b_1, b_2, b_3, b_5, b_6\rangle, & E &= \langle e_1, e_2, e_3, e_4, e_6, e_7\rangle,\\
C &= \langle c_1, c_2, c_3, c_4, c_6, c_7\rangle, & F &= \langle f_1, f_2, f_3, f_4, f_6, f_7\rangle.
\end{aligned}$$

□

Lemma 3.6. *The diagrams below are superhyperbolic for any $\rho > 1$.*

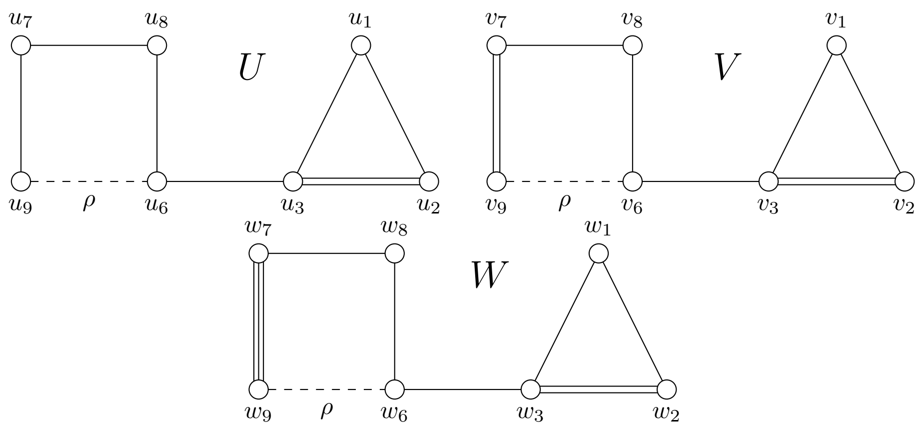


Proof. For $\rho > 1$ we have

$$\begin{aligned}
\det(S_1) &= \frac{1}{16}\Big(4\sqrt{2}\rho^2 - 2\sqrt{2} - 1\Big) > 0,\\
\det(S_2) &= \frac{1}{64}\Big(16\sqrt{2}\rho^2 - 9\sqrt{2} - 6\Big) > 0,\\
\det(S_3) &= \frac{1}{8}\Big(2\sqrt{2}\rho^2 - \sqrt{2} - 1\Big) > 0,\\
\det(S_4) &= \frac{1}{32}\Big(8\sqrt{2}\rho^2 + 4\sqrt{2}\rho - 4\sqrt{2} - 3\Big) > 0,\\
\det(S_5) &= \frac{1}{32}\Big(8\sqrt{2}\rho^2 - 4\sqrt{2} - 3\Big) > 0,\\
\det(S_6) &= \frac{1}{64}\Big(16\sqrt{2}\rho^2 - 9\sqrt{2} - 9\Big) > 0,\\
\det(S_7) &= \frac{1}{64}\Big(16\sqrt{2}\rho^2 + 8\sqrt{2}\rho - 8\sqrt{2} - 9\Big) > 0.
\end{aligned}$$

□

Lemma 3.7. *The diagrams below are superhyperbolic for any $\rho > 1$.*

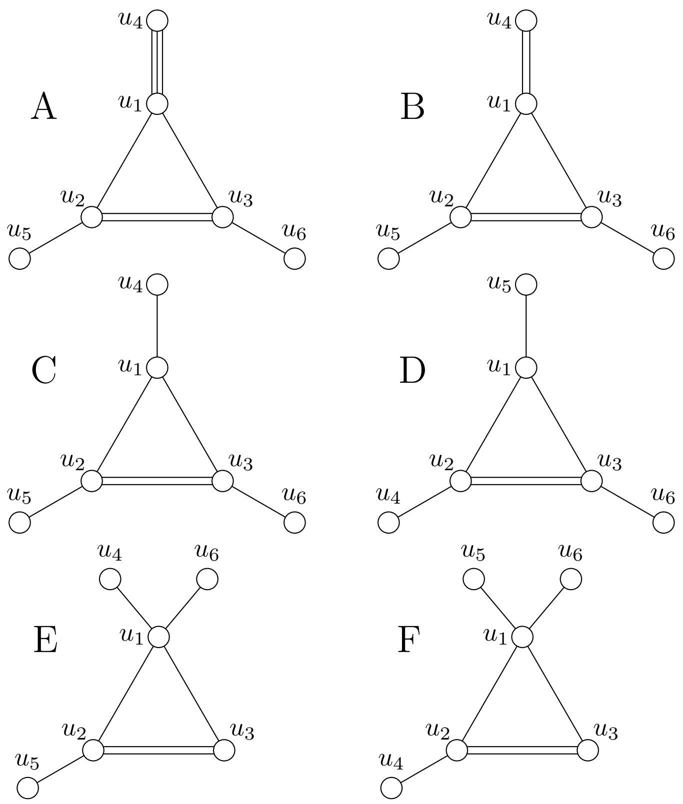


Proof. For $\rho > 1$ we have

$$\begin{aligned}
\det(U) &= \frac{1}{64}\left(12\sqrt{2}\rho^2 + 4\sqrt{2}\rho - 5\sqrt{2} - 6\right) > 0,\\
\det(V) &= \frac{1}{64}\left(12\sqrt{2}\rho^2 + 8\rho - 2\sqrt{2} - 3\right) > 0,\\
\det(W) &= \frac{1}{128}\left(24\sqrt{2}\rho^2 + 4\sqrt{2}(1+\sqrt{5})\rho + 3\sqrt{10} + 3\sqrt{5} - 7\sqrt{2} - 9\right) > 0.
\end{aligned}$$

□

Let us remind that we suppose that $[u_8, L_1] = [u_9, L_1] = 0$, $[u_4, L_1] \geqslant [u_7, L_1]$, and $[u_5, L_1] = [u_6, L_1] = 1$. Thus, the subdiagram $\langle L_1, u_4, u_5, u_6\rangle$ is one of the following.



Case A

$$\begin{aligned}
[u_5, u_4] = [u_5, u_7] = [u_4, u_8] = 0,\\
[u_4, u_6] = [u_4, u_9] = [u_6, u_7] = 0,\\
[u_5, u_6] = [u_5, u_9] = [u_6, u_8] = 0.
\end{aligned}$$

Otherwise, there is either a parabolic or hyperbolic subdiagram that must be elliptic. This implies that $[u_7, u_8] \neq 0$, $[u_8, u_9] \neq 0$, and $[u_9, u_7] \neq 0$. Then the subdiagram $\langle u_7, u_8, u_9\rangle$ is not elliptic.

Case B

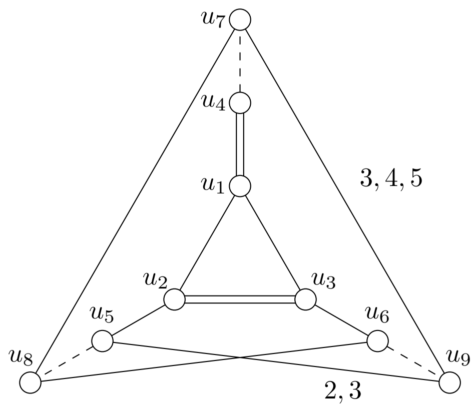
By the same argument we get

$$\begin{aligned}
[u_5, u_4] = [u_5, u_7] = [u_4, u_8] = 0,\\
[u_4, u_6] = [u_4, u_9] = [u_6, u_7] = 0,\\
[u_5, u_6] = 0.
\end{aligned}$$

This yields that, without loss of generality,

$$\begin{gathered}1 \leqslant [u_7, u_8], \quad 1 \leqslant [u_7, u_9] \leqslant 3, \quad [u_8, u_9] = 0, \\ [u_6, u_8] = 1, \quad [u_7, u_8] = 1, \quad [u_5, u_9] \in \{0, 1\}, \\ [u_7, u_1] = [u_7, u_2] = [u_7, u_3] = 0.\end{gathered}$$

Therefore, the diagram is equal to the shown below.



From Lemma 3.7 it follows that the subdiagram $\langle u_1, u_2, u_3, u_6, u_7, u_8, u_9 \rangle$ is superhyperbolic.

Case C

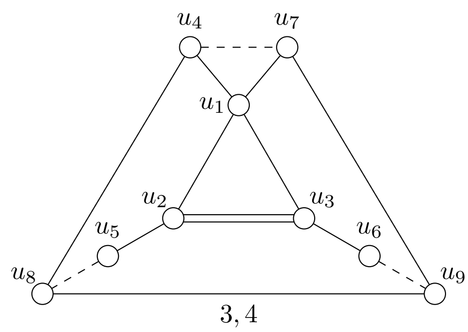
$$[u_5, u_4] = [u_5, u_7] = [u_6, u_4] = [u_6, u_7] = 0.$$

Let $[u_7, u_1] = 0$. Suppose that $[u_8, u_7] = 0$. Then

$$1 \leqslant [u_4, u_8] \leqslant 2, \quad [u_6, u_5] = [u_6, u_8] = 0.$$

Corollary 3.5 (D and E) implies that the diagram $\langle u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8 \rangle$ is superhyperbolic. Therefore, $[u_8, u_7] \geqslant 1$. For similar reasons, $[u_9, u_7] \geqslant 1$. Without loss of generality, $[u_8, u_6] = 1$, $[u_8, u_7] = 1$, and $1 \leqslant [u_9, u_7] \leqslant 3$. The subdiagram $\langle u_1, u_2, u_3, u_6, u_7, u_8, u_9 \rangle$ is superhyperbolic due to Lemma 3.7.

Let $[u_7, u_1] = 1$, then the only possible diagram is shown below.



Corollary 3.5 (A and B) implies that the subdiagram $\langle u_1, u_4, u_5, u_6, u_7, u_8, u_9 \rangle$ is superhyperbolic.

Case D

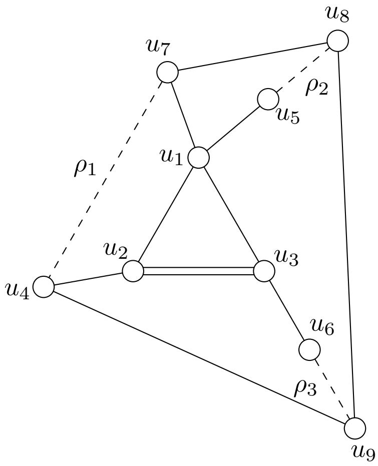
The case $[u_7, L_1] = 0$ is considered in the previous paragraph, so $[u_7, L_1] \neq 0$. Moreover, $[u_7, u_3] = 0$. Suppose that $[u_7, u_2] \geqslant 1$. Then the diagram $\langle L_2, L_3 \rangle$ is not connected. Therefore, $[u_7, u_2] = 0$ and $[u_7, u_1] = 1$. The equality

$$[u_4, u_5] = [u_4, u_8] = [u_7, u_5] = 0$$

implies that $[u_7,u_8]\neq 0$. It is easy to check that

$$[u_4,u_6]=[u_7,u_6]=[u_5,u_6]=[u_8,u_6]=0.$$

Suppose that $[u_5,u_9]\geqslant 1$. Then $[L_2,u_9]=0$ and the subdiagram $\langle L_2,L_4\rangle$ is not connected. Therefore, $[u_5,u_9]=0$, $[u_8,u_9]\geqslant 1$, $[u_7,u_9]=0$, and $[u_4,u_9]\geqslant 1$. The only possible diagram is shown below.



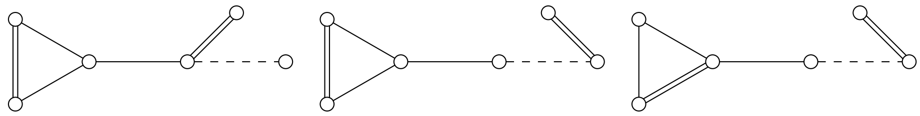
But this diagram is superhyperbolic since

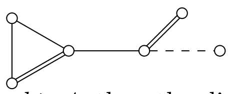
$$\det(\langle u_1,u_2,u_3,u_5,u_7,u_8,u_9\rangle)=\frac{1}{32}\Big(4\big(2\sqrt{2}+1\big)\rho_2^2-4\rho_2-\big(4\sqrt{2}+5\big)\Big)>0$$

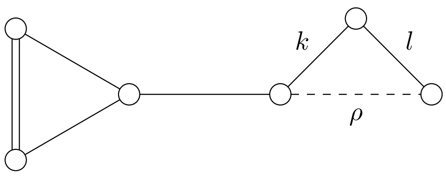
for all $\rho_2>1$.

Case E

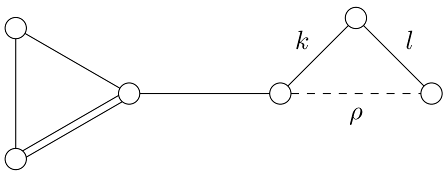
Let $[u_7,L_1]=0$. Lemma 2.15 implies that the diagrams below are superhyperbolic.



The diagram  contains a parabolic subdiagram. Using Remark 2.16, we get that if $k\geqslant 4$ or $l\geqslant 4$, then the diagram below is superhyperbolic for any $\rho>1$.



By the same argument, if $k\geqslant 4$ or $l\geqslant 4$, then the diagram below either contains an unwanted parabolic or Lannér subdiagram or is superhyperbolic for any $\rho>1$.



Therefore, the multiplicity of every edge between the subdiagrams L_2, L_3, and L_4 does no exceed 1.

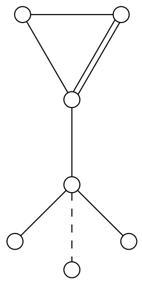
Applying Lemma 3.6 (S_1–S_4) to the subdiagram $\langle u_1, u_2, u_3, u_4, u_7, u_8, u_9\rangle$, we obtain that

$$[u_7, u_8] = [u_4, u_8] = 0 \quad \text{or} \quad [u_7, u_9] = [u_4, u_9] = 0.$$

By the same argument,

$$[u_9, u_8] = [u_6, u_8] = 0 \quad \text{or} \quad [u_9, u_7] = [u_6, u_7] = 0.$$

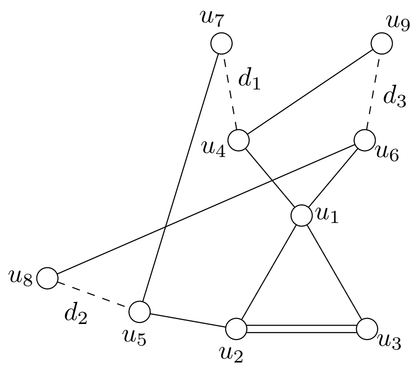
Note that the diagram below contains a parabolic subdiagram.



Thus, applying Lemma 3.6 (S_5–S_7) to the subdiagram $\langle u_1, u_2, u_3, u_5, u_7, u_8, u_9\rangle$, we obtain that

$$[u_8, u_7] = [u_5, u_7] = 0 \quad \text{or} \quad [u_8, u_9] = [u_5, u_9] = 0.$$

It is easy to check that, without loss of generality, the only diagram with such properties is shown below.

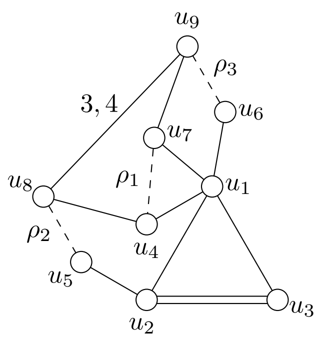


But Corollary 3.5 (F) implies that the subdiagram $\langle u_1, u_2, u_3, u_4, u_6, u_7, u_8, u_9\rangle$ is superhyperbolic.

Let $[u_7, L_1] \neq 0$. Then $[u_7, u_2] = 0$. We also may suppose that $[u_7, u_1] \neq 0$ since $[u_7, u_3] \neq 0$ is already considered in Case D.

$$[u_5, u_4] = [u_5, u_7] = [u_6, u_4] = [u_6, u_7] = 0.$$

Without loss of generality, the only such diagram is shown below.



It is easy to calculate that for $\rho > 1$

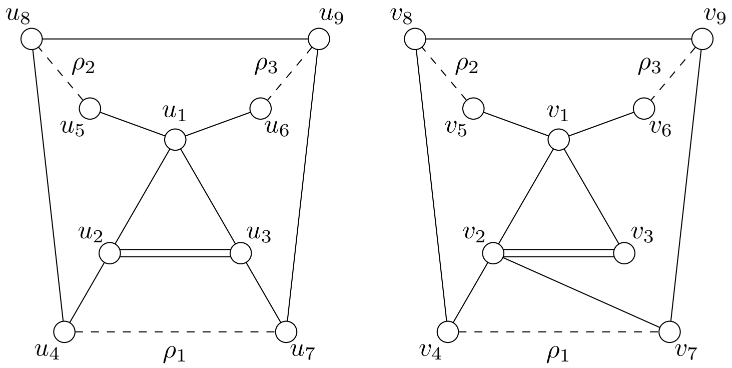
$$\det(\langle u_1, u_2, u_3, u_6, u_7, u_8, u_9\rangle) = \frac{1}{32}\Big(4(1+2\sqrt{2})\rho_3^2 - 4\rho_3 - 4\sqrt{2} - 5\Big) > 0$$

and

$$\det(\langle u_1, u_2, u_3, u_6, u_7, u_8, u_9\rangle) = \frac{1}{32}\Big(4(1+2\sqrt{2})\rho_3^2 - 4\rho_3 - 2\sqrt{2} - 3\Big) > 0.$$

Case F

Let $[u_7, L_1] \neq 0$. The opposite is considered in Case E. The only such diagrams are shown below.



Corollary 3.5 (C and A) implies that $\langle u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9\rangle$ and $\langle v_2, v_4, v_5, v_6, v_7, v_8, v_9\rangle$ are superhyperbolic.

3.2 Proof of Theorem 1.3

We say that a polytope is 3-*free* if every set of facets with an empty intersection contains a pair of disjoint facets. Proposition 2.9 implies that the Coxeter diagram of a compact 3-free Coxeter polytope contains no Lannér subdiagrams of order $\geqslant 3$. Our aim is to prove Theorem 1.3.

The proof is similar to the proof of [Bur22, Theorem 9.4], which is based on the proof of [Vin85, Theorem 6.1]. Thus, we need the Nikulin inequality.

Theorem 3.8 ([Nik81, Theorem 3.2.1])**.** *Let $\theta_0, \dots, \theta_{k-1}$ be non-negative reals, $k \leqslant \lfloor \frac{d}{2} \rfloor$, and P a d-dimensional convex polytope. The following inequality holds*

$$\frac{1}{\alpha_k^P} \sum_{\substack{Q<P \\ \dim Q = k}} \sum_{i=0}^{k-1} \theta_i \alpha_i^Q < \sum_{i=0}^{k-1} \theta_i A_d^{(i,k)},$$

where α_k^R is a number of k-dimensional faces of a polytope R, the notation $Q < P$ means that Q is a face of P, and

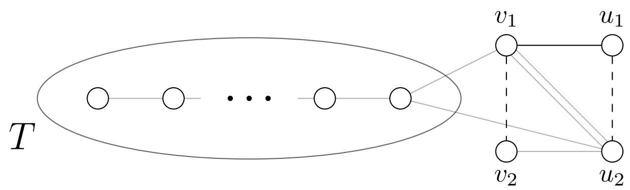
$$A_d^{(i,k)} = \binom{d-i}{k-i} \cdot \frac{\binom{\lceil d/2 \rceil}{i} + \binom{\lfloor d/2 \rfloor}{i}}{\binom{\lceil d/2 \rceil}{k} + \binom{\lfloor d/2 \rfloor}{k}}.$$

Corollary 3.9. *Consider a simple convex d-dimensional polytope, $d \geqslant 3$. The mean edge number of its 2-dimensional faces is less than*

$$A_d^{(1,2)} = \begin{cases} \frac{4(d-1)}{d-2}, & d \text{ is even;} \\ \frac{4d}{d-1}, & d \text{ is odd.} \end{cases}$$

Let $P \subset \mathbb{H}^d$ be a compact Coxeter polytope whose Coxeter diagram S contains no Lannér subdiagrams of order $\geqslant 3$. Denote by a_l the number of its l-dimensional faces and by $a_{2,k}$ the number of its 2-dimensional k-gonal faces. Note that the absence of high-order Lannér subdiagrams implies that $a_{2,3} = 0$.

Lemma 3.10. *Under these assumptions, $a_{2,4} \leqslant a_0 \cdot (d-1)$.*



Proof. Let T be the subdiagram of the diagram S that corresponds to a 4-gonal face (in the sense of Proposition 2.7). There are vertices v_1, v_2, u_1, u_2 of the diagram S such that the diagrams $\langle T, v_i \rangle$, $\langle T, u_i \rangle$, and $\langle T, v_i, u_j \rangle$ are elliptic for $i, j \in \{1, 2\}$ and the diagrams $\langle T, v_1, v_2 \rangle$ and $\langle T, u_1, u_2 \rangle$ are not. Therefore, the subdiagrams $\langle v_1, v_2 \rangle$ and $\langle u_1, u_2 \rangle$ are Lannér diagrams. Indeed, since $\langle T, u_1, u_2 \rangle$ is a hyperbolic diagram, it contains a Lannér subdiagram L which is not contained in either $\langle T, u_1 \rangle$ or $\langle T, u_2 \rangle$. Then L contains both u_1 and u_2. The diagram S contains only Lannér subdiagrams of order 2. Hence, $L = \langle u_1, u_2 \rangle$.

Since the diagram $\langle v_1, v_2, u_1, u_2 \rangle$ is not superhyperbolic, then, without loss of generality, we may assume that $[v_1, u_1] \geqslant 1$. The elliptic diagram $\langle T, v_1, u_1 \rangle$ and its edge $v_1 u_1$ correspond to an angle of the 4-gonal face. The diagram $\langle T, v_1, u_1 \rangle$ is an elliptic subdiagram of order d. Such diagrams correspond to vertices of the polytope P, and there are a_0 of them. Also, elliptic diagrams contain no cycles (Corollary 2.3). Therefore, they contain at most $d-1$ edges.

Thus, every quadrilateral face is associated to at least one pair of an elliptic subdiagram $\langle T, v_1, u_1 \rangle$ of order d and its edge $v_1 u_1$. Since the number of edges contained in the subdiagram does not exceed $d-1$ and the number of such subdiagrams equals a_0, we conclude that $a_{2,4} \leqslant a_0 \cdot (d-1)$. $\square$

Proof of Theorem 1.3. Let $d \geqslant 13$. Assume that there exists a compact hyperbolic Coxeter polytope $P \subset \mathbb{H}^d$ whose Coxeter diagram S contains no Lannér subdiagrams of order $\geqslant 3$. From Corollary 3.9 it follows that the mean number of vertices in 2-dimensional faces $\varkappa = \binom{d}{2} \cdot \frac{a_0}{a_2}$ is less than $\frac{4 \cdot 13}{12} = 4\frac{1}{3}$. Since P contains no 2-dimensional triangular faces,

$$a_{2,4} > \frac{2}{3} \cdot a_2 = \frac{2}{3} \cdot \binom{d}{2} \cdot \frac{a_0}{\varkappa} > \frac{2}{3} \cdot \frac{13 \cdot 12}{2} \cdot \frac{a_0}{13/3} = 12 a_0.$$

On the other hand, Lemma 3.10 implies that $a_{2,4} \leqslant a_0 \cdot (d-1) \leqslant 12 a_0$. $\square$

3.3 Proof of Theorem 1.5

Consider an ideal 7-dimensional right-angled hyperbolic polyhedron P. Let a_k denote the number of its k-dimensional faces (we use f_k only for simplicial polytopes to avoid confusion). The polyhedron P is combinatorially equivalent to a Euclidean polytope. Let us cut off all the vertices of the Euclidean polytope and denote the resulting truncated polytope by P'. Let a'_k denote the number of k-dimensional faces of P'. The following equalities hold:

$$\begin{aligned} a'_0 &= 64 a_0, & a'_1 &= a_1 + 192 a_0, & a'_2 &= a_2 + 240 a_0, \\ a'_3 &= a_3 + 160 a_0, & a'_4 &= a_4 + 60 a_0, & a'_5 &= a_5 + 12 a_0, \\ & & a'_6 &= a_6 + a_0. & & \end{aligned}$$

The polyhedron P is simple at edges. Therefore, the polytope P' is simple, and its dual is simplicial. This fact allows us to apply the Dehn–Sommerville equations:

$$\begin{aligned} a_3 + 160 a_0 &= 5(a_4 + 60 a_0) - 15(a_5 + 12 a_0) + 35(a_6 + a_0) - 70, \\ a_2 + 240 a_0 &= 9(a_4 + 60 a_0) - 34(a_5 + 12 a_0) + 84(a_6 + a_0) - 168, \\ a_1 + 192 a_0 &= 7(a_4 + 60 a_0) - 28(a_5 + 12 a_0) + 70(a_6 + a_0) - 140, \\ 64 a_0 &= 2(a_4 + 60 a_0) - 8(a_5 + 12 a_0) + 20(a_6 + a_0) - 40. \end{aligned}$$

After simplifications we obtain:

$$\begin{aligned}
a_3 &= 5a_4 - 15a_5 + 35a_6 - 5a_0 - 70,\\
a_2 &= 9a_4 - 34a_5 + 84a_6 - 24a_0 - 168,\\
a_1 &= 7a_4 - 28a_5 + 70a_6 - 38a_0 - 140,\\
0 &= 2a_4 - 8a_5 + 20a_6 - 20a_0 - 40.
\end{aligned}$$

Then the average number of vertices in a 2-dimensional face of P equals

$$\begin{aligned}
\varkappa = \frac{192a_0}{a_2} &= \frac{{}^{192}\!/_{20}(2a_4 - 8a_5 + 20a_6 - 40)}{9a_4 - 34a_5 + 84a_6 - 24a_0 - 168} = \\
&= \frac{{}^{192}\!/_{20}(2a_4 - 8a_5 + 20a_6 - 40)}{9a_4 - 34a_5 + 84a_6 - {}^{24}\!/_{20}(2a_4 - 8a_5 + 20a_6 - 40) - 168} = \\
&= \frac{32}{11}\left(1 - \frac{2a_5 - 6a_6 + 12}{a_2}\right).
\end{aligned}$$

We claim that $a_5 > 3a_6$ and therefore $\varkappa < \frac{32}{11} < 3$. The latter readily implies the absence of 7-dimensional hyperbolic ideal right-angled polyhedra since every 2-dimensional face contains at least three vertices.

Indeed, every face of an ideal right-angled hyperbolic polyhedron is itself an ideal right-angled hyperbolic polyhedron. Every 6-dimensional finite volume right-angled hyperbolic polyhedron has at least 27 facets [Duf10, Lemma 1 and Proposition 4]. On the other hand, every 5-dimensional face of P is contained in exactly two 6-dimensional faces of P. Thus, $2a_5 \geqslant 27a_6$.

3.4 Proof of Theorem 1.6

Let P be a finite volume hyperbolic 13-dimensional right-angled polyhedron. Denote the number of its k-dimensional faces by a_k, the number of its finite vertices by v_0, and the number of its ideal vertices by v_∞. Obviously, $a_0 = v_0 + v_\infty$. The polyhedron P is combinatorially equivalent to a Euclidean polytope. Let us cut off all the vertices of the Euclidean polytope that correspond to the ideal vertices of P and denote the obtained polytope by P'. Let a'_k denote the number of k-dimensional faces of P'. The following equalities hold:

$$\begin{array}{lll}
a'_0 = 4096\, v_\infty + v_0, & a'_1 = 24\,576\, v_\infty + a_1, & a'_2 = 67\,584\, v_\infty + a_2,\\
a'_3 = 112\,640\, v_\infty + a_3, & a'_4 = 126\,720\, v_\infty + a_4, & a'_5 = 101\,376\, v_\infty + a_5,\\
a'_6 = 59\,136\, v_\infty + a_6, & a'_7 = 25\,344\, v_\infty + a_7, & a'_8 = 7920\, v_\infty + a_8,\\
a'_9 = 1760\, v_\infty + a_9, & a'_{10} = 264\, v_\infty + a_{10}, & a'_{11} = 24\, v_\infty + a_{11},\\
 & a'_{12} = v_\infty + a_{12}. &
\end{array}$$

The polyhedron P is simple at edges. Therefore, the polytope P' is simple, and its dual is simplicial. This fact allows us to apply the Dehn–Sommerville equations. After simplifications we obtain:

$$\begin{aligned}
a_6 &= -1716a_{12} + 792a_{11} - 330a_{10} + 120a_9 - 36a_8 + 8a_7 - 132v_\infty + 3432,\\
a_5 &= -9009a_{12} + 4125a_{11} - 1683a_{10} + 585a_9 - 159a_8 + 27a_7 - 1089v_\infty + 18\,018,\\
a_4 &= -20\,020a_{12} + 9130a_{11} - 3685a_{10} + 1252a_9 - 325a_8 + 50a_7 - 3740v_\infty + 40\,040,\\
a_3 &= -24\,024a_{12} + 10\,934a_{11} - 4389a_{10} + 1474a_9 - 374a_8 + 55a_7 - 6864v_\infty + 48\,048,\\
a_2 &= -16\,380a_{12} + 7448a_{11} - 2982a_{10} + 996a_9 - 250a_8 + 36a_7 - 7116v_\infty + 32\,760,\\
a_1 &= -6006a_{12} + 2730a_{11} - 1092a_{10} + 364a_9 - 91a_8 + 13a_7 - 3958v_\infty + 12\,012,\\
v_0 &= -924a_{12} + 420a_{11} - 168a_{10} + 56a_9 - 14a_8 + 2a_7 - 924v_\infty + 1848.
\end{aligned}$$

Let $\alpha_0 = \binom{13}{2}$ and $\alpha_\infty = 12 \cdot 2^{11}$ denote the number of 2-faces containing a finite and ideal vertex, respectively. The average number of vertices in a 2-dimensional face of a finite volume right-angled hyperbolic 13-polyhedron is equal to

$$\varkappa = \frac{\alpha_0 v_0 + \alpha_\infty v_\infty}{a_2}.$$

According to Nikulin inequality, $\varkappa < \frac{13}{3}$. Let us consider the following difference:

$$\left(\alpha_0 v_0 + \frac{10309}{6144}\cdot \alpha_\infty v_\infty\right) - \frac{13}{3}\cdot a_2 = 2184 - \frac{26}{3}a_8 + 52a_9 - 182a_{10} + \frac{1456}{3}a_{11} - 1092a_{12}.$$

The coefficients $\frac{13}{3}$ and $\frac{10309}{6144}$ are chosen so in order to cancel a_7 and v_∞ on the right-hand side.

We claim that the difference is negative. Indeed, every face of a hyperbolic finite volume right-angled polyhedron is itself a hyperbolic finite volume right-angled polyhedron. Every 9-dimensional hyperbolic finite volume right-angled polyhedron has at least 152 facets and every 11-dimensional hyperbolic finite volume right-angled polyhedron has at least 564 facets ([Duf10, Lemma 1 and Proposition 4]). However, every 8-face is contained in exactly five 9-faces and every 10-face is contained in exactly three 11-faces. Thus, $152a_9 \leqslant 5a_8$ and $564a_{11} \leqslant 3a_{10}$. Therefore, $52a_9 - \frac{26}{3}a_8 < 0$ and $\frac{1456}{3}a_{11} - 182a_{10} < 0$. Finally, $a_{12} > 2$, so $2184 - 1092a_{12} < 0$.

Thus we proved that

$$\frac{\alpha_0 v_0 + \frac{10309}{6144}\cdot\alpha_\infty v_\infty}{a_2} < \frac{13}{3}.$$

The left part of the inequality is a weighted average number of the vertices in 2-dimensional faces: the contribution of every finite vertex equals 1 and the contribution of every ideal vertex equals $\frac{10309}{6144} > 1.6778$. Meanwhile, $2\cdot\frac{10309}{6144} > 3.3557$ and $3\cdot\frac{10309}{6144} > 5.03369$. Every 2-face of a finite volume right-angled polyhedron is either contains at least 3 ideal vertices, or 2 ideal and 1 finite vertices, or 1 ideal and 3 finite vertices, or 5 finite vertices. Therefore, the weighted average number of the vertices in 2 dimensional face is greater than $4.34 > \frac{13}{3}$. This contradicts the bound we obtained.

3.5 Proof of Theorem 1.8

Recall that $\mathcal{P}^n$ denotes the family of finite volume non-compact right-angled hyperbolic polyhedra, $a_k(P)$ and $v_\infty(P)$ denote the number of k-faces and the number of ideal vertices of a finite volume right-angled hyperbolic polyhedron P respectively. For a polyhedron P denote by $a_k^l(P)$ the average number of l-faces of a k-face. In other words,

$$a_k^l(P) = \frac{1}{a_k(P)}\sum_{\dim F = k} a_l(F),$$

where F runs over all k-faces of P.

Proposition 3.11 ([Nik81], [Kho86, Theorem 10]). *Let P be an n-polytope that is simple at edges. Then*

$$a_k^l(P) < \binom{n-l}{n-k}\frac{\binom{\lceil n/2\rceil}{l} + \binom{\lfloor n/2\rfloor}{l}}{\binom{\lceil n/2\rceil}{k} + \binom{\lfloor n/2\rfloor}{k}}.$$

In [Non15] Nonaka studied the right-angled hyperbolic 3-polyhedra with a single ideal vertex and obtained the following result.

Proposition 3.12 ([Non15, Corollary 3.6]). *If P^3 is a finite volume right-angled hyperbolic 3-polyhedron and $v_\infty(P^3) \leqslant 1$, then $a_2(P^3) \geqslant 12$.*

Corollary 3.13. *If $P^5 \in \mathcal{P}^5$, then $v_\infty(P^5) \geqslant 2$.*

Proof. Suppose that $v_\infty(P^5) \leqslant 1$. Every 3-face of P^5 is a right-angled hyperbolic 3-polyhedron of finite volume with at most one ideal vertex. Therefore, according to Proposition 3.12, every 3-face of P contains at least 12 facets and $a_3^2(P^5) \geqslant 12$. Meanwhile, according to Proposition 3.11, $a_3^2(P^5) < 12$. ∎

Let $\nu(P) = a_{n-1}(P) + v_\infty(P)$ and if $\mathcal{P}^n \neq \varnothing$, let

$$\nu_n = \min_{P^n\in\mathcal{P}^n} \nu(P^n).$$

Dufour used this value to prove Theorem 1.6. He discovered the following relations.

Proposition 3.14 ([Duf10, Proposition 4]). $\nu_5 \geqslant 26$.

Proposition 3.15 ([Duf10, Lemma 1]). *Let $P^n \in \mathcal{P}^n$ with $n \geqslant 3$. Then*

$$a_{n-1}(P^n) \geqslant 1 + \nu_{n-1}.$$

Proposition 3.16 ([Duf10, Lemma 2]). *Let $n \geqslant 3$. Then*

$$\nu_n \geqslant 5 - 2n + 2\nu_{n-1}.$$

Using the double counting technique, one can bound the number of the ideal vertices of a finite volume right-angled hyperbolic polyhedra from below as follows.

Lemma 3.17. *Let $P^n \in \mathcal{P}^n$.*

$$v_\infty(P^n) \geqslant \frac{a_{n-1}(P^n) \cdot v'_\infty(P^n)}{2\,(n-1)},$$

where $v'_\infty(P^n)$ is the minimal number of ideal vertices that a facet of P^n contains.

Proof. Since every ideal vertex of the polytope P^n is contained in exactly $2\,(n-1)$ facets, the following inequality holds:

$$v_\infty(P^n) \cdot 2\,(n-1) \;=\; \sum_{\dim F = n-1} v_\infty(F) \;\geqslant\; a_{n-1}(P^n) \cdot v'_\infty(P^n),$$

where F runs over all facets of P^n. ☐

Let $P^n \in \mathcal{P}^n$. Proposition 3.14 and Proposition 3.16 imply that $\nu_5 \geqslant 26$, $\nu_6 \geqslant 45$, and $\nu_7 \geqslant 81$. Proposition 3.15, Corollary 3.13, and Lemma 3.17 provide

$$\begin{aligned}
a_5(P^6) &\geqslant 27, &\quad v_\infty(P^6) &\geqslant \left\lceil \frac{27 \cdot 2}{10} \right\rceil = 6,\\
a_6(P^7) &\geqslant 46, &\quad v_\infty(P^7) &\geqslant \left\lceil \frac{46 \cdot 6}{12} \right\rceil = 23,\\
a_7(P^8) &\geqslant 82, &\quad v_\infty(P^8) &\geqslant \left\lceil \frac{82 \cdot 23}{14} \right\rceil = 135.
\end{aligned}$$

Now applying the definition of ν_n, Proposition 3.15, and Lemma 3.17 we obtain

$$\begin{aligned}
\nu_8 &\geqslant 217, & a_8\,(P^9) &\geqslant 218, & v_\infty(P^9) &\geqslant 1704,\\
\nu_9 &\geqslant 1922, & a_9\,(P^{10}) &\geqslant 1923, & v_\infty(P^{10}) &\geqslant 182\,044,\\
\nu_{10} &\geqslant 183\,967, & a_{10}\,(P^{11}) &\geqslant 183\,968, & v_\infty(P^{11}) &\geqslant 1\,674\,504\,428,\\
\nu_{11} &\geqslant 1\,674\,688\,396, & a_{11}\,(P^{12}) &\geqslant 1\,674\,688\,397, & v_\infty(P^{12}) &\geqslant 127\,466\,960\,740\,760\,088.
\end{aligned}$$

3.6 Proof of Theorem 1.9

The Lobachevsky function is concave on the interval $[0, \frac{\pi}{2}]$ which implies that

$$\sum_{k=1}^{m} \Lambda\,(x_k) \leqslant m\,\Lambda\left(\frac{\sum_{k=1}^{m} x_k}{m}\right).$$

Let P be an ideal right-angled hyperbolic polyhedron and v a vertex of P, which will further be called an *apex*. For every face f there is a unique projection u of point v to f. The projection will lie on the interior of f unless f meets one of the faces containing v. Projecting u to the edges of f will decompose P into tetrahedra known as orthoschemes: a hyperbolic tetrahedron with vertices P_1, P_2, P_3, and P_4 is said to be an *orthoscheme* if edge P_1P_2 is orthogonal to plain $P_2P_3P_4$, and $P_1P_2P_3$ is orthogonal to P_3P_4. Such a decomposition for the face formed by the vertices v_1, v_2, v_3, and v_4 is shown in Figure 3.1.

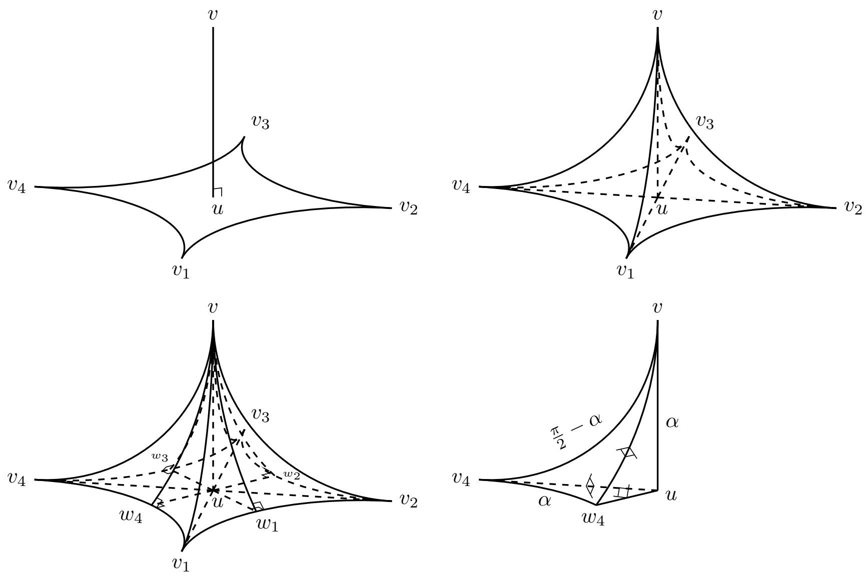


Figure 3.1: Decomposition of an ideal right-angled polyhedron.

Thus, we get eight tetrahedra having a common edge vu. Consider a tetrahedron formed by v, u, w_4, and v_4, where vertices v and v_4 are ideal, and vertices u and w_4 are finite. Dihedral angles at edges vw_4, w_4u and uv_4 equal $\pi/2$. If dihedral angle at vu equals α, then dihedral angle at vv_4 equals $\pi/2-\alpha$ and dihedral angle at v_4w_4 equals to α. Thus, this tetrahedron is determined by α, and we call α a *parameter* of the tetrahedron. By [Thu80, Chapter 7], volume of the tetrahedron formed by v, u, w_4, and v_4 equals $\frac{1}{2}\Lambda(\alpha)$.

Example 3.18. Let P be the antiprism $A(4)$ with the vertex set $V = \{v_1, v_2, \ldots, v_8\}$. A decomposition of the $A(4)$ with apex at v_1 is shown in Figure 3.2. Let us define a *tetrahedral cone* $C(v)$ of the vertex v as the union of tetrahedra of a decomposition containing v. Therefore, $A(4)$ splits in cones and $\mathrm{Vol}(A(4)) = \sum_{k=2}^{8} \mathrm{Vol}(C(v_k))$.

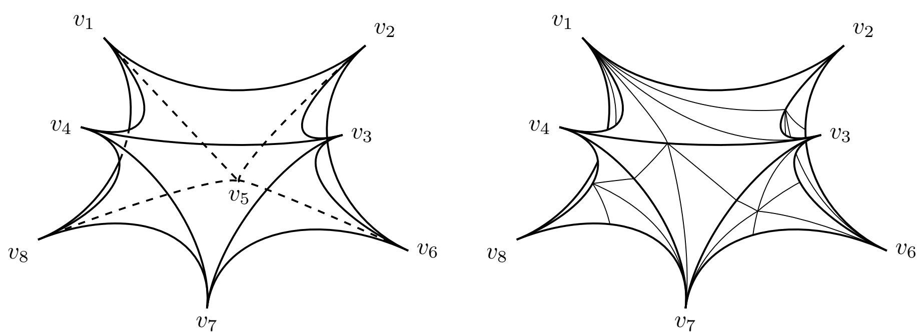


Figure 3.2: Antiprism $A(4)$ and its decomposition.

Vertices v_2, v_4, v_5, and v_8 are adjacent to v_1. The cone $C(v_4)$ consists of two tetrahedra with parameters α and β. Since the dihedral angle at the edge v_1v_4 is $\pi/2$, then the sum $(\pi/2-\alpha) + (\pi/2-\beta)$ equals $\pi/2$ and $\alpha+\beta$ equals $\pi/2$. The concavity of $\Lambda(x)$ implies

$$\mathrm{Vol}(C(v_4)) = \frac{1}{2}\Lambda(\alpha) + \frac{1}{2}\Lambda(\beta) \leqslant \Lambda\left(\frac{\pi}{4}\right).$$

The same holds for every vertex adjacent to v_1. For more details see [Atk09, Prop. 5.2].

A similar argument provides $\mathrm{Vol}(C(v)) \leqslant 2\Lambda(\pi/4)$ if v is quasi-adjacent to v_1 (i.e., v_3), and

$\mathrm{Vol}(C(v)) \leqslant 4\Lambda(\pi/4)$ for every other v (i.e., v_6 and v_7). So

$$\mathrm{Vol}(A(4)) \leqslant 4 \cdot \Lambda\left(\frac{\pi}{4}\right) + 1 \cdot 2\Lambda\left(\frac{\pi}{4}\right) + 2 \cdot 4\Lambda\left(\frac{\pi}{4}\right) = 14\Lambda\left(\frac{\pi}{4}\right).$$

Example 3.18 shows the idea of the proof of the following lemma.

Lemma 3.19. *Let P be an ideal right-angled hyperbolic polyhedron with V vertices. If there is a vertex with m quasi-adjacent vertices, then*

$$\mathrm{Vol}(P) \leqslant \left(V - 4 - \frac{m}{2}\right) \cdot \frac{v_8}{2}.$$

Proof. Every vertex has exactly four adjacent vertices. This fact provides

$$\mathrm{Vol}(P) \leqslant (V - 1 - 4 - m) \cdot 4\Lambda\left(\frac{\pi}{4}\right) + m \cdot 2\Lambda\left(\frac{\pi}{4}\right) + 4 \cdot \Lambda\left(\frac{\pi}{4}\right) = \left(V - 4 - \frac{m}{2}\right) \cdot 4\Lambda\left(\frac{\pi}{4}\right).$$

The lemma is proved. ☐

To prove item (3) of Theorem 1.9 we shall sum up the volumes of the tetrahedra not over the vertices but over the faces. A *tetrahedral cone* $C(f)$ of the face f is a union of the decomposition tetrahedra having a part of f as a face. Such a cone is shown at the bottom left in Figure 3.1.

Lemma 3.20. *Let P be a decomposed ideal right-angled hyperbolic polyhedron and f a k-gonal face of P that does not contain the apex.*

(1) If f is quasi-incident to the apex, then

$$\mathrm{Vol}(C(f)) \leqslant (k-1) \cdot \Lambda\left(\frac{\pi}{2k-2}\right).$$

(2) If f is not quasi-incident to the apex, then $\mathrm{Vol}(C(f)) \leqslant k\,\Lambda\left(\frac{\pi}{k}\right)$.

Proof. If f is quasi-incident to the apex, then the projection of the apex does not lie inside f and the cone $C(f)$ contains $2k-2$ tetrahedra of the decomposition (like $v_3v_4v_7$ in Figure 3.2) and

$$\mathrm{Vol}(C(f)) = \sum_{i=2}^{2k-1} \frac{1}{2}\Lambda(\alpha_i) \leqslant (k-1) \cdot \Lambda\left(\frac{\pi}{2k-2}\right), \quad \text{where } \sum_{i=2}^{2k-1} \alpha_i = \pi.$$

If f is not quasi-incident to the apex, then the projection of the apex lies inside f and the cone $C(f)$ contains $2k$ tetrahedra of the decomposition (like $v_3v_6v_7$ in Figure 3.2) and

$$\mathrm{Vol}(C(f)) = \sum_{i=1}^{2k} \frac{1}{2}\Lambda(\alpha_i) \leqslant k \cdot \Lambda\left(\frac{\pi}{k}\right), \quad \text{where } \sum_{i=1}^{2k} \alpha_i = 2\pi.$$

The lemma is proved. ☐

Now let us prove Theorem 1.9.

Proof. Lemma 3.19, Lemma 2.22, and Remark 2.23 provide items (1) and (2) of Theorem 1.9.

Let us prove item (3). Let P be an ideal right-angled hyperbolic polyhedron with $V \geqslant 73$ vertices and F faces that are only triangular and quadrilateral. By Proposition 2.24 there is a vertex quasi-incident to 8 pairwise different quadrilaterals. Then by Lemma 3.20

$$\begin{aligned}\mathrm{Vol}(P) \leqslant 8 \cdot 3\Lambda\left(\frac{\pi}{3}\right) + 8 \cdot (4-1)\Lambda\left(\frac{\pi}{2 \cdot 4 - 2}\right) + (F - 20) \cdot 4\Lambda\left(\frac{\pi}{4}\right) &= \\ = (V - 18) \cdot 4\Lambda\left(\frac{\pi}{4}\right) + 24 \cdot \left[\Lambda\left(\frac{\pi}{3}\right) + \Lambda\left(\frac{\pi}{6}\right)\right]&.\end{aligned}$$

To complete the proof we recall that $v_8 = 8\Lambda(\pi/4)$ and $v_3 = 3\Lambda(\pi/3) = 2\Lambda(\pi/6)$ (since $\Lambda(x)$ is an odd function, π-periodic and $\Lambda(2x) = 2\Lambda(x) + 2\Lambda(x + \frac{\pi}{2})$, see [Thu80, Lemma 7.1.4]). ☐

3.7 Proof of Theorem 1.10

3.7.1 Proof of statement (1)

Let us use the enumeration of faces as in Figure 2.4, with f_1 and f_3 containing e. Since $V > 80$, then by Corollary 2.27 there is an edge $e \in P$ such that for k_i-gonal faces f_i, $i = 1, \ldots, 4$ we have $\sum_{i=1}^{4} k_i \geqslant 24$.

Let P' be a polyhedron obtained by gluing P with its image under reflection in the plane passing through the face f_1. Then P' has $V' = 2V - 2k_1$ vertices. Denote by f_2' a $(2k_2 - 4)$-gonal face of P' containing f_2, by f_3' a $(2k_3 - 4)$-gonal face of P' containing f_3, and by f_4' a $(2k_4 - 4)$-gonal face of P' containing f_4.

Recall the following volume bound from [EV20b].

Lemma 3.21 ([EV20b, Corollary 3.2]). *Let P be a compact right-angled hyperbolic 3-polyhedron with V vertices. Let f_1, f_2, and f_3 be three faces of P such that f_2 is adjacent to both f_1 and f_3, and f_i is k_i-gonal for $i = 1, 2, 3$. Then the following formula holds:*

$$\mathrm{Vol}(P) \leqslant (V - k_1 - k_2 - k_3 + 4) \cdot \frac{5v_3}{8}.$$

Applying Lemma 3.21 to the polyhedron P' we get

$$2\mathrm{Vol}(P) = \mathrm{Vol}(P') \leqslant (2V - 2k_1 - (2k_2 - 4) - (2k_3 - 4) - (2k_4 - 4) + 4) \cdot \frac{5v_3}{8},$$

whence

$$\mathrm{Vol}(P) \leqslant (V - k_1 - k_2 - k_3 - k_4 + 8) \cdot \frac{5v_3}{8} \leqslant (V - 16) \cdot \frac{5v_3}{8} = \frac{5v_3}{8} \cdot V - 10v_3,$$

where we used inequality $\sum_{i=1}^{4} k_i \geqslant 24$. Statement (1) is proved,

3.7.2 Proof of statement (2)

Denote $k_1 = k$. Since any face of P has at least 5 sides, we have $k_i \geqslant 5$ for $i = 2, 3, 4$. Then, from preceding inequality we get

$$\mathrm{Vol}(P) \leqslant (V - k - 5 - 5 - 5 + 8) \cdot \frac{5v_3}{8} = \frac{5v_3}{8} \cdot V - \frac{5k + 35}{8} v_3.$$

Thus, Theorem 1.10 is proved.

3.8 Proof of Theorem 1.11

Observe that formula (1.1) implies that the following upper bound holds for a polyhedron P with $V_\infty > 0$ ideal and V_F finite vertices:

$$\mathrm{Vol}(P) < \frac{v_8}{2} \cdot V_\infty + \frac{5v_3}{8} \cdot V_F - \frac{v_8}{2}.$$

Let P^1 denote the polyhedron P, and let k_∞^1, resp. k_F^1, be the number of ideal, resp. finite, vertices of a face f^1 of P^1. Consider P^2 the union of P^1 with its image under the reflection is the plane containing the face f^1. Then P^2 is right-angled with $V_\infty^2 = 2V_\infty - k_\infty^1$ ideal vertices and $V_F^2 = 2V_F - 2k_F^1$ finite vertices. Then applying to P^2 the upper bound from (1.1), we get

$$\mathrm{Vol}(P) = \frac{\mathrm{Vol}(P^2)}{2} < \frac{v_8}{2} \cdot V_\infty + \frac{5v_3}{8} \cdot V_F - \left(\frac{v_8}{4} \cdot k_\infty^1 + \frac{5v_3}{8} \cdot k_F^1 + \frac{v_8}{4} \right) =$$
$$= \frac{v_8}{2} \cdot V_\infty + \frac{5v_3}{8} \cdot V_F - c_1 - \frac{v_8}{4},$$

where

$$c_1 = \frac{v_8}{4} \cdot k_\infty^1 + \frac{5v_3}{8} \cdot k_F^1.$$

Let k_∞^2, resp. k_F^2, be the number of ideal, resp. finite, vertices of a face f^2 of P^2. Consider P^3 the union of P^2 with its image under the reflection in the plane containing the face f^2. Then P^3 is right-angled with $V_\infty^3 = 4V_\infty - 2k_\infty^1 - k_\infty^2$ ideal vertices and $V_F^3 = 4V_F - 4k_F^1 - 2k_F^2$ finite vertices. Applying to P^3 the upper bound from (1.1), we get

$$\mathrm{Vol}(P) = \frac{\mathrm{Vol}(P^3)}{4} < \frac{v_8}{2} \cdot V_\infty + \frac{5v_3}{8} \cdot V_F - c_1 - c_2 - \frac{v_8}{8},$$

where

$$c_2 = \frac{v_8}{8} \cdot k_\infty^2 + \frac{5v_3}{16} \cdot k_F^2.$$

Continuing the process inductively, we obtain

$$\mathrm{Vol}(P) < \frac{v_8}{2} \cdot V_\infty + \frac{5v_3}{8} \cdot V_F - c_1 - c_2 - \ldots - c_n - \frac{v_8}{2^{n+1}}, \tag{3.4}$$

where for $i = 1, \ldots, n$ we have

$$c_i = \frac{v_8}{2^{i+1}} \cdot k_\infty^i + \frac{5v_3}{2^{i+2}} \cdot k_F^i.$$

Suppose that for every $i = 1, 2, \ldots,$ the face f^i is *not* an ideal triangle and has at least 6 neighbouring faces. It will be demonstrated in Lemma 3.22 below that one can choose such a face f^i of P^i for every i. Then the value c_i is minimal for $k_\infty^i = 2$ and $k_F^i = 2$. Thus, we get that

$$\sum_{i=1}^{n} c_i \geqslant \left(v_8 + \frac{5v_3}{2}\right) \cdot \left(\sum_{i=1}^{n} \frac{1}{2^i}\right) = \left(v_8 + \frac{5v_3}{2}\right) \cdot \left(1 - \frac{1}{2^n}\right).$$

Taking the limit $n \to \infty$ in (3.4), we obtain

$$\mathrm{Vol}(P) \leqslant \frac{v_8}{2} \cdot V_\infty + \frac{5v_3}{8} \cdot V_F - \left(v_8 + \frac{5v_3}{2}\right).$$

Thus, the bound from Theorem 1.11 is obtained. It remains to prove Lemma 3.22.

Let $N_6(P)$ denote the set of faces of a polyhedron P with the following property: $f \in N_6(P)$ if f has at least 6 neighbouring faces. This set is non-empty by part (1) of Lemma 2.28.

Lemma 3.22. *If $V_\infty + V_F > 17$ then for any $i = 1, \ldots, n$ the set $N_6\left(P^i\right)$ contains at least one face that is not an ideal triangle.*

Proof. Let us observe that for any $i \geqslant 1$ the polyhedron P^i is not an octahedron. Indeed, this holds true for $P^1 = P$ since $V_\infty^1 = V_\infty$, $V_F^1 = V_F$ and $V_\infty^1 + V_F^1 > 17$. Assume by a contradiction that P^2 is an octahedron. Then $2V_\infty^1 \geqslant V_\infty^2 = 6$ and

$$\mathrm{Vol}\left(P^2\right) = 2 \cdot \mathrm{Vol}\left(P^1\right) \geqslant 2 \cdot \frac{4V_\infty^1 + V_F^1 - 8}{32} \cdot v_8 > \frac{4V_\infty^1 + 17 - V_\infty^1 - 8}{16} \cdot v_8 \geqslant \frac{18 \cdot v_8}{16} > v_8,$$

which is a contradiction. Finally, let us show that for any $i \geqslant 3$ polyhedron P^i is not an octahedron. If we assume by a contradiction that P^i is an octahedron, then it would hold that $V_\infty^1 \geqslant 1$ and by inequality (1.1) we would obtain

$$\begin{aligned}\mathrm{Vol}\left(P^i\right) \geqslant 2^{i-1} \cdot \frac{4V_\infty^1 + V_F^1 - 8}{32} \cdot v_8 &> \\ > 2^{i-6} \cdot (4V_\infty^1 + 17 - V_\infty^1 - 8) \cdot v_8 &\geqslant \\ \geqslant 12 \cdot 2^{i-6} \cdot v_8 = \frac{3}{2} \cdot 2^{i-3} \cdot v_8 &> v_8,\end{aligned}$$

which is again a contradiction.

Let $i = 1$ and assume by a contradiction that all faces from $N_6(P^1)$ are ideal triangles. Then part (2) of Lemma 2.28 implies that $N_6(P^1)$ contains at least 7 ideal triangles. Denote the set of the faces of P^1 by $\mathcal{F}$ and the number of the ideal vertices of a face f by $I(f)$. Suppose that P^1 contains at most 5 ideal vertices. Then we get a contradiction:

$$21 = 3 \cdot 7 \leqslant \sum_{f \in \mathcal{F}} I(f) \leqslant 4 \cdot 5 = 20.$$

Thus, P^1 contains at least 6 ideal vertices.

Since P^1 is not an octahedron, there is a face f' that is not an ideal triangle. Therefore, $f' \notin N_6(P^1)$, whence f' has at most 5 neighbouring faces. Then by part (3) of Lemma 2.28 there is a face f'' which has at least 7 neighbouring faces. Therefore, f'' is not an ideal triangle. However $f'' \in N_6(P^1)$, which contradicts the assumption.

Now let $i \geqslant 2$ and assume by a contradiction that for some $i \geqslant 2$ each face from $N_6\left(P^i\right)$ is an ideal triangle. The polyhedron P^i is the union of two copies of P^{i-1} along the face f^{i-1}. Let D^{i-1} be the set of faces of P^{i-1} that have a common edge with f^{i-1}. Let S^i denote the set of such faces of P^i that contain a face from D^{i-1}. That is, S^i consists of all of the new faces that appeared after the union of two copies of P^{i-1} along f^{i-1}. By Theorem 2.17, each face of a right-angled polyhedron has at least 5 neighbours. Hence each face from the set S^i has at least 6 neighbours and $S^i \subset N_6(P^i)$. Therefore, by our assumption, each face from S^i is an ideal triangle. Then each face from D^{i-1} is a triangle with two ideal and one finite vertices. Moreover, f^{i-1} is a face with an even number of vertices, such that its ideal and finite vertices alternate among themselves. Namely, if f^{i-1} has $2k$ vertices, then there are k ideal and k finite vertices, and $k \geqslant 2$.

Observe that there are at least 2 ideal vertices in P^{i-1} that are not contained in the face f^{i-1}. Indeed, since all faces from D^{i-1} are triangles with 2 ideal vertices, then P^{i-1} has at least one ideal vertex v that is not contained in the face f^{i-1}. Suppose that v is the only such vertex. Then v is incident to all of the vertices of f^{i-1}. If $k \geqslant 3$, then v is a vertex of valency $2k$. The latter is impossible by Theorem 2.17. If $k = 2$ then P^{i-1} is a quadrilateral pyramid with 5 faces, while by Theorem 2.17 the quadrilateral pyramid is not a right-angled polytope. Thus, P^{i-1} should have at least 2 ideal vertices that are not contained in the face f^{i-1}.

Hence P^i has at least $k + 4 \geqslant 6$ ideal vertices. Since P^i is not an octahedron, it has at least one face f' that is not an ideal triangle. Using part (3) of Lemma 2.28 we obtain that there is a face f'' of P^i that has at least 7 neighbours and thus cannot be an ideal triangle. This contradicts our assumption about $N_6\left(P^i\right)$, and the lemma is proved. $\square$

Author's papers

[ABVE23] S. A. Alexandrov, N. V. Bogachev, A. Yu. Vesnin, and A. A. Egorov. On volumes of hyperbolic right-angled polyhedra. *Mat. Sb.*, 214:3–22, 2023.

[Ale22] S. Alexandrov. Lannér diagrams and combinatorial properties of compact hyperbolic Coxeter polytopes. 2022. Preprint available at arXiv:2203.07248 (accepted to *Trans. Amer. Math. Soc.*).

[Ale23] S. Alexandrov. On ideal vertices of right-angled hyperbolic polyhedra. 2023. Preprint available at arXiv:2303.09533.

Bibliography

[All06] D. Allcock. Infinitely many hyperbolic Coxeter groups through dimension 19. *Geom. Topol.*, 10:737–758, 2006.

[All13] D. Allcock. Reflection centralizers in Coxeter groups. *Transf. Groups*, 18(3):599–613, 2013.

[And70a] E. M. Andreev. On convex polyhedra in Lobachevskii spaces. *Math. USSR Sbornik*, 10:413–440, 1970.

[And70b] E. M. Andreev. On convex polyhedra of finite volume in Lobachevskii spaces. *Math. USSR Sbornik*, 12:255–259, 1970.

[Atk09] C. Atkinson. Volume estimates for equiangular hyperbolic Coxeter polyhedra. *Algebr. Geom. Topol.*, 9:1225–1254, 2009.

[BBKS21] M. Belolipetsky, N. Bogachev, A. Kolpakov, and L. Slavich. Subspace stabilisers in hyperbolic lattices. 2021. arXiv:2105.06897.

[Bel16] M. Belolipetsky. Arithmetic hyperbolic reflection groups. *Bull. (New Series) of the Am. Maths. Soc.*, 53(3):437–475, 2016.

[Bel21] G. Belletti. The maximum volume of hyperbolic polyhedra. *Trans. Amer. Math. Soc.*, 374:1125–1153, 2021.

[BK21] N. Bogachev and A. Kolpakov. On faces of quasi-arithmetic Coxeter polytopes. *Int. Math. Res. Notices*, 2021:3078–3096, 2021.

[Bog17] N. V. Bogachev. Reflective anisotropic hyperbolic lattices of rank 4. *Uspekhi Mat. Nauk*, 72:193–194, 2017.

[Bog19] N. V. Bogachev. Classification of (1,2)$-$reflective anisotropic hyperbolic lattices of rank 4. *Izvestiya Math.*, 83:1–19, 2019.

[Bog20] N. Bogachev. From geometry to arithmetic of compact hyperbolic Coxeter polytopes. *To appear in Transf. Groups*, 2020. arXiv:2003.11944.

[Bor87] R. Borcherds. Automorphism groups of Lorentzian lattices. *Journal of Algebra*, 111:133–153, 1987.

[BP16] V. M. Buchstaber and T. E. Panov. On manifolds defined by 4-colourings of simple 3-polytopes. *Russian Math. Surveys*, 71:1137–1139, 2016.

[BP18] N. Bogachev and A. Perepechko. Vinberg's algorithm for hyperbolic lattices. *Math. Notes*, 103(5):836–840, 2018.

[Bug84] V. O. Bugaenko. Groups of automorphisms of unimodular hyperbolic quadratic forms over the ring $\mathbf{Z}\left[\frac{\sqrt{5}+1}{2}\right]$. *Vestnik Moskov. Univ. Ser. 1. Mat. Mekh.*, pages 6–12, 1984.

[Bug92] V. O. Bugaenko. Arithmetic crystallographic groups generated by reflections, and reflective hyperbolic lattices. *Advances in Soviet Mathematics*, 8:33–55, 1992.

[Bur22] A. Burcroff. Near classification of compact hyperbolic Coxeter d-polytopes with $d+4$ facets and related dimension bounds. 2022. arXiv:2201.03437.

[CKP22] A. Champanerkar, I. Kofman, and J. Purcell. Right-angled polyhedra and alternating links. *Algebr. Geom. Topol.*, 22:739–784, 2022.

[Cox34] H. S. M. Coxeter. Discrete groups generated by reflections. *Ann. Math.*, 35(3):588–621, 1934.

[DJ91] M. Davis and T. Januszkievicz. Convex polytopes, Coxeter orbifolds and torus actions. *Duke Math. J.*, 62:417–451, 1991.

[Duf10] G. Dufour. Notes on right-angled coxeter polyhedra in hyperbolic spaces. *Geom. Dedicata*, 147:277–282, 2010.

[Ess96] F. Esselmann. The classification of compact hyperbolic Coxeter d-polytopes with $d+2$ facets. *Comment. Math. Helvetici*, 71:229–242, 1996.

[EV20a] A. Egorov and A. Vesnin. Ideal right-angled polyhedra in Lobachevsky space. *Chebyshevskii Sbornik*, 21:65–83, 2020.

[EV20b] A. Egorov and A. Vesnin. On correlation of hyperbolic volumes of fullerenes with their properties. *Comput. Math. Biophys.*, 8:150–167, 2020.

[EV20c] A. Egorov and A. Vesnin. Volume estimates for right-angled hyperbolic polyhedra. *Rend. Istit. Mat. Univ. Trieste*, 52:565–576, 2020.

[FT08] A. Felikson and P. Tumarkin. On hyperbolic Coxeter polytopes with mutually intersecting facets. *J. Combin. Theory A*, 115:121–146, 2008.

[FT09] A. Felikson and P. Tumarkin. Coxeter polytopes with a unique pair of non-intersecting facets. *J. Combin. Theory A*, 116:875–902, 2009.

[Ino22] T. Inoue. Exploring the list of smallest right-angled hyperbolic polyhedra. *Experimental Mathematics*, 31:165–183, 2022.

[JT18] M. Jacquemet and S. Tschantz. All hyperbolic Coxeter n-cubes. *J. Combin. Theory A*, 158:387–406, 2018.

[Kap74] I. M. Kaplinskaya. Discrete groups generated by reflections in the faces of simplicial prisms in Lobachevskian spaces. *Mat. Zametki*, 15:159–164, 1974.

[Kel22] R. Kellerhals. A polyhedral approach to the arithmetic and geometry of hyperbolic chain link complements. Preprint available at `https://homeweb.unifr.ch/kellerha/pub/Kellerhals.pdf`, 2022.

[Kho86] A. G. Khovanskii. Hyperplane sections of polyhedra, toroidal manifolds and discrete groups in Lobachevskii space. *Funct. Anal. Appl.*, 20:41–50, 1986.

[KMT15] A. Kolpakov, B. Martelli, and S. Tschantz. Some hyperbolic three-manifolds that bound geometrically. *Proc. Amer. Math. Soc.*, 143:4103–4111, 2015.

[Kol12] A. Kolpakov. On the optimality of the ideal right angled 24-cell. *Algebr. Geom. Topol.*, 12(4):1941–1960, 2012.

[KV78] I. M. Kaplinskaya and E. B. Vinberg. On the groups $O_{18,1}(\mathbf{Z})$ and $O_{19,1}(\mathbf{Z})$. *Soviet Math. Dokl.*, 238:1273–1275, 1978.

[Lan50] F. Lannér. On complexes with transitive groups of automorphisms. *Comm. Sém. Math. Univ. Lund*, 11:1–71, 1950.

[MMT20] J. S. Meyer, C. Millichap, and R. Trapp. Arithmeticity and hidden symmetries of fully augmented pretzel link complements. *New York J. Math.*, 26:149–183, 2020.

[MZ22a] J. Ma and F. Zheng. Compact hyperbolic Coxeter 4-polytopes with 8 facets. 2022. arXiv:2201.00154.

[MZ22b] J. Ma and F. Zheng. Compact hyperbolic Coxeter five-dimensional polytopes with nine facets. 2022. arXiv:2203.16049.

[Nik81] V. V. Nikulin. On the classification of arithmetic groups generated by reflections in Lobachevsky spaces. *Izv. Akad. Nauk SSSR Ser. Mat.*, 45:113–142, 1981.

[Non15] J. Nonaka. The number of cusps of right-angled polyhedra in hyperbolic spaces. *Tokyo J. Math.*, 38(2):539–560, 2015.

[Poi82] H. Poincaré. Théorie des groupes fuchsiens. *Acta Math.*, 1:1–62, 1882.

[Pro86] M. N. Prokhorov. The absence of discrete reflection groups with noncompact fundamental polyhedron of finite volume in Lobachevskii space of large dimension. *Izv. Akad. Nauk SSSR Ser. Mat.*, 50(2):413–424, 1986.

[PV05] L. Potyagailo and E. B. Vinberg. On right-angled reflection groups in hyperbolic spaces. *Comment. Math. Helvetici*, 80:63–73, 2005.

[RHD07] R. K. W. Roeder, J. H. Hubbard, and W. D. Dunbar. Andreev's theorem on hyperbolic polyhedra. *Ann. Inst. Fourier*, 57:825–882, 2007.

[Thu80] W. Thurston. *The Geometry and Topology of Three-Manifolds*. Lecture notes, Princeton. 1980.

[Tum07] P. Tumarkin. Compact hyperbolic Coxeter n-polytopes with $n+3$ facets. *Electron. J. Combin.*, 14, 2007.

[Ves98] A. Vesnin. Volumes of hyperbolic Löbell 3-manifolds. *Math. Notes*, 64:15–19, 1998.

[Ves17] A. Vesnin. Right-angled polyhedra and hyperbolic 3-manifolds. *Russian Math. Surveys*, 72:335–374, 2017.

[Vin72] E. B. Vinberg. On groups of units of certain quadratic forms. *Mat. Sb.*, 87:18–36, 1972.

[Vin84] E. B. Vinberg. Absence of crystallographic groups of reflections in Lobachevsky spaces of large dimension. *Tr. Mosk. Mat. Obs.*, 47:68–102, 1984.

[Vin85] E. B. Vinberg. Hyperbolic reflection groups. *Uspekhi Mat. Nauk*, 40:29–66, 1985.

[Vin93a] E. B. Vinberg, editor. *Geometry II. Spaces of constant curvature*, volume 29 of *Encyclopaedia of Mathematical Sciences*. Springer Berlin Heidelberg, 1993.

[Vin93b] E. B. Vinberg. Volumes of non-Euclidean polyhedra. *Russian Math. Surveys*, 48:15–45, 1993.